Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 05 July 2019

1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u} + x^3 u\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) + u(1) = 3$.

   (b) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) - u(1) = 3$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^3 + \sin u + |x-1|^{3/2}, \qquad u(0) = u(2) = 5.$$

3. Let us set, for every $\lambda > 0$,

$$I(\lambda) := \inf\left\{\int_0^4 \left(\dot{u}^4 + \dot{u}^2 - \lambda \sin(\dot{u}^2) + xu^2\right)\, dx : u \in C^1([0,4]),\ u(0) = u(4) = 0\right\}.$$

   (a) Determine for which values of λ it turns out that $I(\lambda)$ is a real number.

   (b) Determine for which values of λ it turns out that $I(\lambda)$ is actually a minimum.

   (c) Determine the limit of $I(\lambda)$ as $\lambda \to +\infty$.

4. For every real number $m > 0$, let us set

$$J(m) := \inf\left\{\int_0^1 \left(u^{19} + \arctan(u^2)\right)\, dx : u \in C^1([0,1]),\ u(0) = 0,\ \int_0^1 |\dot{u}|^7\, dx \le m\right\}.$$

   (a) Determine for which values of m it turns out that $J(m)$ is a real number.

   (b) Determine whether there exists $m > 0$ such that $J(m) = 0$.

   (c) Determine for which real values of α it turns out that

$$\lim_{m \to +\infty} \frac{J(m)}{m^\alpha} = 0.$$

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.