Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 11 June 2019

1. Let us consider the functional

$$F(u) = \int_0^\pi \left(\dot{u}^2 - u \sin x\right) \, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_0^\pi u(x)\, dx = 0$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^7 - x^7, \qquad u(0) = 7, \quad u'(7) = 7.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left\{\dot{u}^2 + \dot{u}^5 - \sin(u^2) + u^5\right\} \, dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Determine for which values of ℓ the infimum is actually a minimum.

4. Let us set, for every $\varepsilon > 0$,

$$m_\varepsilon := \inf \left\{ \int_0^4 \left(\dot{u}^2 - u \sin(u^2)\right) \, dx : u \in C^1([0, 4]),\ u(0) = 0,\ u(4) = \varepsilon \right\}.$$

   (a) Determine for which values of ε the infimum is actually a minimum.

   (b) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

   (c) Compute the leading term of m_ε as $\varepsilon \to +\infty$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.