1. Let us consider the functional

$$F(u) = \int_{-1}^{1} (\ddot{u}^2 + \dot{u}^2)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) = u'(0) = 1$.

(b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

$$\delta F(u,v) = 2\int (\ddot{u}\,\ddot{v} + \dot{u}\,\dot{v})\,dx = 2\int (u^{(4)} - \ddot{u})v + [\ddot{u}\dot{v}] - [(\dddot{u} - \dot{u})v]$$

Since the conditions are given in an interior point, we split the interval

$[0,1]$		$[-1,0]$	
$u^{(4)} = \ddot{u}$	ELE	$u^{(4)} = \ddot{u}$	ELE
$u(0) = \dot{u}(0) = 1$	given BCs	$u(0) = \dot{u}(0) = 1$	given BCs
$\ddot{u}(1) = 0$	new BCs	$\ddot{u}(-1) = 0$	new BCs
$\dddot{u}(1) = \dot{u}(1)$		$\dddot{u}(-1) = \dot{u}(-1)$	

$$\boxed{u_1(x) = \frac{1}{e^2+1}e^{x} - \frac{e^2}{e^2+1}e^{-x} + \frac{2e^2}{e^2+1} \qquad u_2(x) = \frac{e^2}{e^2+1}e^{x} - \frac{1}{e^2+1}e^{-x} + \frac{2}{e^2+1}}$$

The sol. $u_1(x)$ and $u_2(x)$ glue in a C^1, but not C^2, way. Their union is the unique min. point in $H^2((-1,1))$, due to standard convexity arguments

$$F(u+v) = F(u) + \delta F(u,v) + F(v) \quad \text{for every admissible } u, v.$$

In more regular classes, from C^2 to C^∞, the min does not exist, and the infimum coincides with the min in H^2.

(b) The key observation is the following: if $u(x)$ is any competitor for part (b), then $u(x) + 1 - u(0)$ is a competitor for part (a) with the same value of $F(u)$.

(and any competitor for part (a) is a competitor for part (b))

Therefore, part (b) admits a min in $H^2((-1,1))$, and the set of minimum points is of the form

$$u_0(x) + c \qquad c \in \mathbb{R}$$

where $u_0(x)$ is the unique min point for part (a).

Again the min in C^2 or more regular classes does not exist.

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2. Let us consider the boundary value problem

$$u''(x) = \frac{1+e^{u(x)}}{1+e^{u'(x)}}, \qquad u(0)=3, \quad u(3)=0.$$

(a) Discuss existence, uniqueness and regularity of the solution.

(b) Prove that $u'(0) < -1$.

$$\ddot{u}(1+e^{\dot{u}}) = 1+e^{u} \; ; \; (\dot{u}+e^{\dot{u}})' = 1+e^{u}$$

$$F(u) = \int_0^3 \left\{ \frac{1}{2}\dot{u}^2 + e^{\dot{u}} + u + e^{u} \right\} dx$$

$$L(x,s,p) = \underbrace{\frac{1}{2}p^2 + e^{p}}_{\text{convex}} + s + e^{s}$$

(a) • Weak formulation in $H^1((0,3))$. We need to allow the value $+\infty$ and observe that $s+e^s$ has a "negative" growth of order 1, and therefore the integral is well-defined.

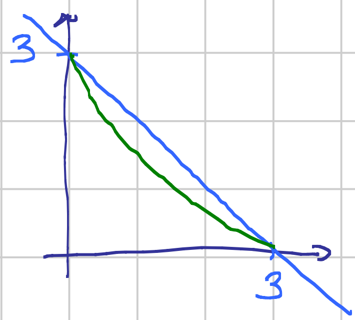
• Compactness: from $F(u) \leq M$ we can deduce that $\|\dot{u}\|_{L^2} \leq M'$ and $\|u\|_{L^\infty} \leq M''$ due to the DBCs. Again it is essential that

$$L(x,s,p) \geq \underset{\text{order 2}}{\frac{1}{2}p^2} - \underset{\text{order 1}}{|s|}$$

• LSC: standard due to convexity wrt p and continuity wrt s (here we need $u_n \to u_\infty$ uniformly).

• Regularity. We can obtain ELE in weak form (observe that we can differentiate $F(u+tv)$), from which we deduce that $\dot{u}+e^{\dot{u}} \in H^1$. Now we observe that $p+e^p$ has a smooth inverse and we proceed by bootstrap.

(b) The solution is strictly convex, and therefore from the BCs it follows that $u(x) < 3-x$ for every $x \in (0,1)$. This inequality implies that $\dot{u}(0) < -1$ (note that $\dot{u}(x)$ is strictly increasing and $\dot{u}(c) = -1$ for some $c \in (0,3)$).



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3. Let us set, for every $\ell > 0$,

$$I(\ell) := \inf\left\{\int_0^\ell \left(\dot{u}^2 - x\sin^2 u\right)\,dx : u \in C_c^\infty((0,\ell))\right\}.$$

(a) Determine whether there exist positive values of ℓ such that $I(\ell) = 0$.

(b) Determine whether there exist positive values of ℓ such that $I(\ell) < 0$.

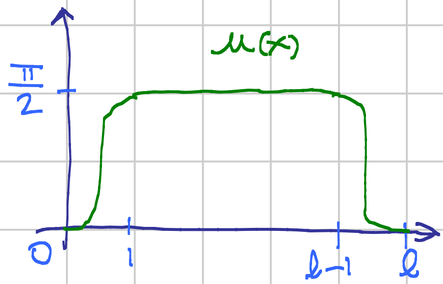
(a) $\boxed{\text{YES}}$ For $\ell = 1$. Indeed it turns out that

$$\int_0^\ell (\dot{u}^2 - x\sin^2 u) \geq \int_0^\ell (\dot{u}^2 - \sin^2 u) \geq \int_0^\ell (\dot{u}^2 - u^2) \underset{\text{true if } \ell \leq \pi}{\geq 0}$$

for every $u \in C_c^\infty((0,\ell))$

(The same argument works for every $\ell < \pi$)

(b) $\boxed{\text{YES}}$ Let us consider the function $u(x)$ as in the figure



The integral $\int_0^\ell \dot{u}(x)^2 dx$ does not depend on ℓ.

The integral $\int_0^\ell x\sin^2 u(x)\, dx$ grows linearly with ℓ.

Therefore $F(u) < 0$ if ℓ is large enough, and therefore the infimum is negative.

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4. Let us set, for every $\varepsilon > 0$,

$$m_\varepsilon := \inf \left\{ \int_0^{2\pi} \left(\varepsilon \ddot{u}^2 + \sin \dot{u} + \cos u \right) dx : u \in C^2([0,1]),\ u(0) = 0,\ u(2\pi) = 2\pi \right\}.$$

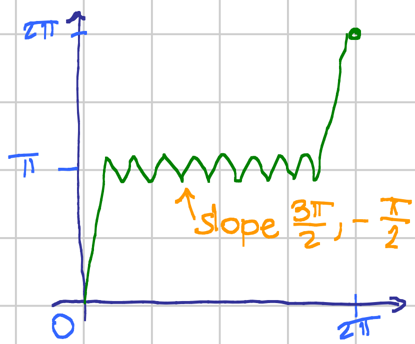
(a) Determine for which positive values of ε it turns out that m_ε is a minimum.

(b) Compute the limit of m_ε as $\varepsilon \to 0^+$.

(a) $\boxed{m_\varepsilon \text{ is a minimum for every } \varepsilon > 0}$ The standard direct method works.

- <u>Compactness</u> Since $\varepsilon \ddot{u}^2 + \sin \dot{u} + \cos u \geq \varepsilon \ddot{u}^2 - 2$, an estimate such as $F(u) \leq M$ implies $\|\ddot{u}\|_{L^2} \leq M'$.
  By Rolle's theorem there exists $c \in (0, 2\pi)$ such that $\dot{u}(c) = 1$, and therefore we obtain $\|\dot{u}\|_{L^\infty} \leq M''$.
  At this point the DBCs imply that $\|u\|_{L^\infty} \leq M'''$
- <u>LSC</u> : standard, due to the convexity wrt $\ddot{u}$ and the continuity and boundedness from below wrt $\dot{u}$ and u
- <u>Regularity</u>. The weak form of ELE is enough to deduce that $u \in C^2$. Further regularity (not required) can be proved by bootstrap.

(b) $\boxed{m_\varepsilon \to -4\pi}$ It is trivial that $m_\varepsilon \geq 4\pi$ for every $\varepsilon > 0$.



The main idea is that

$$\Gamma\text{-}\limsup F_\varepsilon(u) \leq \int_0^{2\pi} (\sin \dot{u} + \cos u) = F_0(u)$$

(just use constant recovery sequences), and therefore

relaxation

$$\Gamma\text{-}\limsup F_\varepsilon(u) \leq \overline{F_0}(u) = \underbrace{\int_0^{2\pi} (-1 + \cos u)}$$

the minimum of this functional is -4π

regardless of the BCs. Details are left to the interested reader. ☺

The figure above shows a possible behavior of a minimizing sequence.

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