Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 23 February 2019

1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\ddot{u}^2 + \dot{u}^2\right) \, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) = u'(0) = 1$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Let us consider the boundary value problem

$$u''(x) = \frac{1 + e^{u(x)}}{1 + e^{u'(x)}}, \qquad u(0) = 3, \quad u(3) = 0.$$

   (a) Discuss existence, uniqueness and regularity of the solution.

   (b) Prove that $u'(0) < -1$.

3. Let us set, for every $\ell > 0$,

$$I(\ell) := \inf \left\{ \int_0^\ell \left(\dot{u}^2 - x \sin^2 u\right) \, dx : u \in C_c^\infty((0, \ell)) \right\}.$$

   (a) Determine whether there exist positive values of ℓ such that $I(\ell) = 0$.

   (b) Determine whether there exist positive values of ℓ such that $I(\ell) < 0$.

4. Let us set, for every $\varepsilon > 0$,

$$m_\varepsilon := \inf \left\{ \int_0^{2\pi} \left(\varepsilon \ddot{u}^2 + \sin \dot{u} + \cos u\right) \, dx : u \in C^2([0, 1]), \; u(0) = 0, \; u(2\pi) = 2\pi \right\}.$$

   (a) Determine for which positive values of ε it turns out that m_ε is a minimum.

   (b) Compute the limit of m_ε as $\varepsilon \to 0^+$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.