1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - 3u\dot{u} + xu\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u(1) = 0$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

ELE: $(2\dot{u} - 3u)' = -3\dot{u} + x$, $\quad 2\ddot{u} - \cancel{3\dot{u}} = -\cancel{3\dot{u}} + x$, $\quad \ddot{u} = \frac{1}{2}x$

General solution is $\boxed{u(x) = \frac{1}{12}x^3 + a + bx}$

(a) $u(0) = 0 \rightsquigarrow a = 0$

$u(1) = 0 \rightsquigarrow \frac{1}{12} + b = 0 \rightsquigarrow b = -\frac{1}{12}$

$\boxed{u_0(x) = \frac{1}{12}(x^3 - x)}$ is the unique min. point. Indeed

$$F(u_0 + v) = F(u_0) + \underbrace{\int_0^1 (2\dot{u}_0 - 3u_0)\dot{v} + (x - 3\dot{u}_0)v}_{\delta F(u_0, v) = 0} + \int_0^1 \dot{v}^2 - 3v\dot{v}$$

and the last term is

$$\int_0^1 \dot{v}^2 - 3v\dot{v} = \int_0^1 \dot{v}^2 - \underbrace{\frac{3}{2}\left[v^2\right]_{x=0}^{x=1}}_{=0} \geq 0$$

with $=$ if and only if $v \equiv$ const $\equiv 0$ because of DBC.

(b) $\boxed{\text{Inf} = -\infty}$ and a minimizing sequence is

$u_n(x) = nx \quad$ because

$$F(u_n) = \int_0^1 n^2 - 3n^2 x + nx^2 \longrightarrow -\infty$$

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2. Let us consider ordinary differential equation

$$\ddot{u} = \frac{(7+\sin x)u}{7+\sin \dot{u}}.$$

(a) Prove that the equation admits a 2π-periodic solution.

(b) Prove that every 4π-periodic solution is actually 2π-periodic.

$\ddot{u}(7+\sin\dot{u}) = (7+\sin x)u$, $\quad (7\dot{u} - \cos\dot{u})' = (7+\sin x)u$

$$F(u) = \int_0^{2\pi} \left\{ \frac{7}{2}\dot{u}^2 - \sin\dot{u} + \frac{7+\sin x}{2} u^2 \right\} dx$$

$L(x,s,p) := \frac{7}{2}p^2 - \sin p + \frac{7+\sin x}{2} s$ is strictly convex in (s,p)

$L_p|_{x=0} = L_p|_{x=2\pi} \Leftrightarrow \dot{u}(0) = \dot{u}(2\pi)$ (L_p strictly monotone)

- Variational formulation : u is a 2π-periodic solution of the equ if and only if u is a minimum point for $F(u)$ with $u(0)=u(2\pi)$. Indeed any min. point satisfies also $\dot{u}(0) = \dot{u}(2\pi)$, and since the RHS is 2π periodic as well, $u(x+2\pi)$ coincides with $u(x)$.

- Compactness We observe that $L(x,s,p) \geq \dot{u}^2 + u^2 - A$, and therefore from $F(u) \leq M$ we deduce $\|\dot{u}\|_{L^2} \leq M'$ and $\|u\|^2 \leq M''$.
- LSC : follows in a standard way because L is convex wrt p and continuous and bounded from below wrt s.
- Uniqueness : follows from strict convexity
- Regularity : standard argument (ELE in weak form + bootstrap).

(b) As before, every 4π-periodic solution is a minimum point of

$$\min\{F(u) : u \in H^1((0,4\pi)),\ u(0) = u(4\pi)\}$$

The minimum point is unique, and solves ELE with PBC in $[0,4\pi]$. But we know a solution of this equation, namely the solution to point (a) repeated twice.

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left(\sqrt{1+\dot{u}^4}-\sqrt{1+u^2}\right)dx : u \in C^1([0,\ell]),\ u(0)=u(\ell)=0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(b) Determine for which values of ℓ the infimum is a real number.

(a) $\boxed{u_0(x) \equiv 0 \text{ is never a SLM}}$ Indeed the second variation is

$$\delta^2 F(u_0, v) = -\int_0^\ell v^2\,dx \quad < 0 \qquad \text{if } v \neq 0.$$

(b) $\boxed{\text{The infimum is a real number for every } \ell > 0}$ and actually the inf is a min. This follow by the direct method.

- Weak formulation in $H^1((0,\ell))$. We observe that the Lagrangian satisfies

$$L(x,s,p) \geq p^2 - s - 1$$

and therefore the integral is well defined

- Compactness : due to the previous estimate, from $F(u) \leq M$ we deduce that $\|\dot{u}\|_{L^2} \leq M'$ (here it is essential that the growth wrt p is larger than the "negative growth" wrt s), and then from the DBCs we deduce that $\|u\|_{L^\infty} \leq M''$.
- LSC : standard because of the convexity of L wrt p and because of the uniform convergence $u_n \to u_\infty$
- Regularity. Any min. point satisfies ELE in weak form. From the strict monotonicity of L_p this is enough to guarantee that the min. point is of class C^1.

Remark It is not possible to obtain further regularity by bootstrap (but further regularity is not required). The reason is that L_p does NOT admit a smooth inverse function.

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4. Let us set

$$m_\varepsilon = \inf \left\{ \int_0^1 \left(\varepsilon \dot{u}^2 + \dot{u}^4 - \sin(u^2) + u^4 \right) dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon \right\},$$

and

$$M_\varepsilon = \inf \left\{ \int_0^1 \left(\varepsilon \dot{u}^2 + \dot{u}^4 - \sin(u^{22}) + u^4 \right) dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon \right\}.$$

For every real number $\alpha > 0$, compute the following limits:

$$\lim_{\varepsilon \to 0^+} \frac{m_\varepsilon}{\varepsilon^\alpha}, \qquad \lim_{\varepsilon \to 0^+} \frac{M_\varepsilon}{\varepsilon^\alpha}.$$

(a) $\boxed{\frac{m_\varepsilon}{\varepsilon^\alpha} \to -\infty \text{ for every } \alpha > 0}$ This is because

$$m_\varepsilon \to m_0 = \min \left\{ \int_0^1 (\dot{u}^4 - \sin u^2 + u^4) : u(0) = u(1) = 0 \right\} < 0$$

The minimum is negative because $u_0(x) \equiv 0$ is not even a DLM because the second variation is $\int -v^2 < 0$. Existence of the min. follows in a standard way from the direct method.

In order to prove the convergence, it is enough to observe that

- $m_\varepsilon \geq m_0$ for every $\varepsilon > 0$
- $F_\varepsilon(u) \to F_0(u)$ for every $u \in C^1$.

(b) $\boxed{m_\varepsilon = +\varepsilon^3 + o(\varepsilon^3)}$ To begin with, we observe that

$$s^4 - \sin(s^{22}) \geq 0 \quad \forall s \in \mathbb{R}$$

It follows that

$$m_\varepsilon \geq \min \left\{ \int_0^1 \varepsilon \dot{u}^2 : u(0) = 0,\ \dot{u}(0) = \varepsilon \right\} = \varepsilon^3$$

On the other hand, it turns out that

$$F_\varepsilon(\varepsilon x) = \varepsilon^3 + o(\varepsilon^3),$$

from which the conclusion follows.

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