Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 02 February 2019

1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - 3u\dot{u} + xu\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u(1) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

2. Let us consider the square $Q := (0, \pi)^2$ in the plane.

   Find all exponents $p \geq 1$ for which there exists a constant C_p such that

$$\int_0^\pi [f(t, \sin t)]^2\, dt \leq C_p \left\{ \int_Q \|\nabla f(x,y)\|^p\, dx\, dy \right\}^{2/p} \qquad \forall f \in C_c^\infty(Q).$$

3. Let $\Omega \subseteq \mathbb{R}^2$ be the unit ball with center in $(4, 5)$. For every real number λ, let us set

$$I(\lambda) = \inf \left\{ \int_\Omega \left(\arctan y \cdot u_x^2 + \arctan x \cdot u_y^2 - \lambda \frac{u^4}{1+u^2} \right) dx\, dy : u \in C_c^\infty(\Omega) \right\}.$$

   (a) Determine whether there exists $\lambda > 0$ such that $I(\lambda)$ is a real number.

   (b) Determine whether there exists $\lambda > 0$ such that $I(\lambda) = -\infty$.

4. For every sequence $\{x_n\}_{n\geq 1}$, let us set

$$T(x_1, x_2, x_3, \ldots, x_n, \ldots) = \left(\frac{x_1}{\sqrt{1}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \ldots, \frac{x_n}{\sqrt{n}}, \ldots \right).$$

   (a) Determine whether the restriction of T defines a continuous operator for each of the following choices of the sequence space:

$$\ell^2 \to \ell^2, \qquad \ell^2 \to \ell^1, \qquad \ell^3 \to \ell^2.$$

   When the answer is positive, determine the norm of the operator.

   (b) Determine for which values of the exponent $p \geq 1$ the restriction of T defines a continuous operator $\ell^p \to \ell^1$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.