1. Let us consider the functional

$$F(u) = \int_0^\pi \left[(\dot{u} - x^2)^2 + \sin x \cdot u\right]\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u'(\pi) = \pi$.

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ELE: $\quad 2(\dot{u} - x^2)' = \sin x, \quad 2\ddot{u} - 4x = \sin x, \quad \ddot{u} = 2x + \frac{1}{2}\sin x$

$$\rightsquigarrow \boxed{u(x) = \frac{1}{3}x^3 - \frac{1}{2}\sin x + a + bx}$$

(a) BCs: $\quad u(0) = 0 \rightsquigarrow a = 0$

$\dot{u} - x^2 \big|_{x=\pi} = 0 \rightsquigarrow \dot{u}(\pi) = \pi^2 \rightsquigarrow x^2 - \frac{1}{2}\cos x + b = \pi^2$ for $x = \pi$

$\rightsquigarrow \pi^2 - \frac{1}{2}\cos\pi + b = \pi^2 \Rightarrow b = -\frac{1}{2}$

$\rightsquigarrow \boxed{u_0(x) = -\frac{1}{2}\sin x + \frac{1}{3}x^3 - \frac{1}{2}x}$ is the unique min. point because

$$F(u_0 + v) = F(u_0) + \underbrace{\int_0^\pi \{2\dot{v}(\dot{u}_0 - x^2) + \sin x \cdot v\}}_{=0 \text{ because of ELE}} + \underbrace{\int_0^\pi \dot{v}^2}_{\geq 0}$$

The last term is ≥ 0, and $= 0$ iff $v \equiv$ constant, but constant $= 0$ because of the DBC in $x = 0$.

(b) $\boxed{\operatorname{Inf} = -\infty}$ A minimizing sequence is

$$u_n(x) = \pi x - n.$$

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2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{1+u^3+x^2}{1+\dot{u}^2}, \qquad u(0) = u'(3) = 3.$$

$$\ddot{u}(1+\dot{u}^2) = 1+u^3+x^2 \qquad \left(\dot{u}+\frac{1}{3}\dot{u}^3+c\right)' = 1+u^3+x^2 \qquad c=-12$$

$$F(u) = \int_0^3 \left\{\frac{1}{2}\dot{u}^2 + \frac{1}{12}\dot{u}^4 - 12\dot{u} + u + \frac{1}{4}u^4 + x^2 u\right\} dx$$

$$L(x,s,p) = \frac{1}{2}p^2 + \frac{1}{12}p^4 - 12p + s + \frac{1}{4}s^4 + x^2 s$$

NBC : $L_p = 0 \iff \underbrace{p+\frac{1}{3}p^3}_{\text{injective}} = 12 \iff p=3$

$L(x,s,p)$ is strictly convex in (s,p) and therefore u satisfies the equ. if and only if u is a min. point of $F(u)$ in $H^1((0,3))$.

- weak formulation in $H^1((0,3))$ with $u(0)=3$. We allow the value $+\infty$.
- Compactness. Follows from the estimates

$$\frac{1}{2}p^2 + \frac{1}{12}p^4 - 12p \geq \frac{1}{2}p^2 - A \qquad \forall p \in \mathbb{R}$$

$$s + \frac{1}{4}s^4 + x^2 s \geq \frac{1}{4}s^4 - 10|s| \geq s^2 - B \qquad \forall s \in \mathbb{R} \quad \forall x \in [0,3]$$

here we use DBC

Thus from $F(u) \leq M$ we obtain $\|\dot{u}\|_{L^2} \leq M'$, $\|u\|_{L^2} \leq M''$, $\|u\|_{L^\infty} \leq M'''$

- LSC : standard because L is convex wrt p and continuous and bounded from below wrt s.

- Uniqueness : strict convexity of L in (s,p).

- Regularity. The minimizer u satisfies ELE (why?) in the form

$$\int_0^3 \left(\dot{u}+\frac{1}{3}\dot{u}^3-12\right)\dot{v} + (1+u^3+x^2)v = 0 \qquad \forall v \in C_c^\infty((0,3))$$

and therefore

$1+u^3+x^2$ is the weak derivative of $\dot{u}+\frac{1}{3}\dot{u}^3-12$. But the function $\psi(p) = p+\frac{1}{3}p^3$ has an inverse of class C^∞ ...

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^{\ell} \left(\sqrt{1+\dot{u}^2} - \sqrt{1+u^4} \right) dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine for which values of ℓ the infimum is a real number.

(a) $u_0(x) \equiv 0$ satisfies ELE. The second variation is

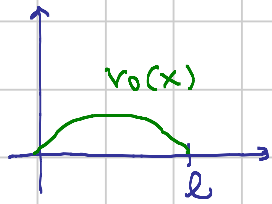
$$\delta^2 F(u_0, v) = \int_0^{\ell} \dot{v}^2 \, dx$$ which is clearly strictly positive.

Therefore $u_0(x) \equiv 0$ is $\boxed{\text{WLM for every } \ell > 0}$

(b) The excess $E(x, s, p, q) \geq 0$ because L is convex wrt p.

Therefore $u_0(x) \equiv 0$ is $\boxed{\text{SLM for every } \ell > 0}$

(c) $\boxed{\text{Inf} = -\infty \text{ for every } \ell > 0}$



If $v_0(x)$ is as in the figure, then

$$F(n v_0) = \int_0^{\ell} \underbrace{\sqrt{1+n^2 \dot{v}_0^2}}_{\sim n} - \underbrace{\sqrt{1+n^4 v_0^4}}_{\sim n^2} \rightarrow -\infty$$

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4. Let us consider the functional

$$F(u) = \int_0^1 \{\sin \dot{u} + \sin(u - \sin x)\}\, dx.$$

(a) Compute the infimum of $F(u)$ in the class $C_c^\infty((0,1))$.

(b) Is is true that any minimizing sequence for the previous point converges in some sense to a continuous function?

(a) The infimum of F is equal to the infimum of the relaxed functional (for example in $L^2((0,1))$).

The relaxation involves the convex envelope of $\sin p$, which is -1. The boundary conditions are lost in the limit. Therefore

$$\overline{F}(u) = \int_0^1 \{-1 + \sin(u - \sin x)\}\, dx$$

$\boxed{\text{The infimum is } -2}$

(b) $\boxed{\text{It is NOT TRUE}}$ Let us consider the function

$$u_0(x) = \begin{cases} \frac{3\pi}{2} + \sin x & \text{if } x \in (0, \frac{1}{2}) \\ \frac{7\pi}{2} + \sin x & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

Then it turns out that $\overline{F}(u_0) = -2$ and therefore $u_0(x)$ can be approximated in many senses, for example also uniform convergence on compact subsets of $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, by a sequence $\{u_m\} \subseteq C_c^\infty((0,1))$ such that

$$F(u_m) \to -2.$$

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