Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 15 January 2019

1. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{1+u^3+x^2}{1+\dot{u}^2}, \qquad u(0) = u'(3) = 3.$$

2. Let V denote the set of sequences $\{x_n\}_{n\geq 1}$ of real numbers such that

$$\sum_{n=1}^{\infty} n|x_n| < +\infty,$$

with norm defined by the series above.

   (a) Characterize the dual space of V.

   (b) Determine all aligned functionals of the sequence with $x_1 = 9$, $x_2 = 8$, $x_3 = 7$, and $x_n = 0$ for every $n \geq 4$.

   (c) Determine all aligned functionals of the sequence with $x_n = (-1)^n \cdot n^{-4}$.

   (A aligned functional of a vector v is a linear 1-Lipschitz functional f such that $f(v) = \|v\|$)

3. Let $\Omega = (-1, 1)^2$ be a square in the plane.

   (a) Determine whether

$$\sup\left\{\int_\Omega u_{xy}^2\,dx\,dy : u \in C_c^2(\Omega),\ \int_\Omega u_{xx}^2\,dx\,dy \leq 7,\ \int_\Omega u_{yy}^2\,dx\,dy \leq 8\right\}$$

   is finite or infinite.

   (b) Determine whether

$$\sup\left\{\int_\Omega u_{yy}^2\,dx\,dy : u \in C_c^2(\Omega),\ \int_\Omega u_{xx}^2\,dx\,dy \leq 7,\ \int_\Omega u_{xy}^2\,dx\,dy \leq 8\right\}$$

   is finite or infinite.

4. Let us consider the open set $\Omega = (0, \pi)^2$. Determine if there exists a constant C such that

$$\int_\Omega \sin(xy) \cdot u^2\,dx\,dy \leq C \int_\Omega \left(e^{xy} \cdot u_x^2 + u_y^2 + \cos x \cdot u_x \cdot u_y\right)\,dx\,dy$$

for every $u \in C_c^1(\Omega)$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.