Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 15 January 2019

1. Let us consider the functional

$$F(u) = \int_0^\pi \left[(\dot{u} - x^2)^2 + \sin x \cdot u\right] \, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u'(\pi) = \pi$.

2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{1 + u^3 + x^2}{1 + \dot{u}^2}, \qquad u(0) = u'(3) = 3.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left(\sqrt{1 + \dot{u}^2} - \sqrt{1 + u^4} \right) dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Determine for which values of ℓ the infimum is a real number.

4. Let us consider the functional

$$F(u) = \int_0^1 \{\sin \dot{u} + \sin(u - \sin x)\} \, dx.$$

   (a) Compute the infimum of $F(u)$ in the class $C_c^\infty((0, 1))$.

   (b) Is is true that any minimizing sequence for the previous point converges in some sense to a continuous function?

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.