1. Let us consider the functional

$$F(u) = \int_0^2 \left(\dot{u}^2 + (u - x^2)^2\right)\, dx.$$

Discuss the minimum problem for $F(u)$ with each of the following sets of boundary conditions:

(a) $u(0) = 0$ and $u(2) = 4$.

(b) $u(0) = 0$ and $u'(2) = 4$.

(a) In a standard way we obtain ELE with BCs

$$\begin{cases} \ddot{u} = u - x^2 \\ u(0) = 0 \\ u(2) = 4 \end{cases}$$

$\rightsquigarrow$ general solution is

$$u(x) = a \cosh x + b \sinh x + x^2 + 2$$

$\rightsquigarrow$ $a = -2$ $\quad b = \dfrac{2(\cosh 2 - 1)}{\sinh 2}$

$$\boxed{u_0(x) = -2\cosh x + \frac{2(\cosh 2 - 1)}{\sinh 2} + x^2 + 2}$$

is the unique minimizer due to the strict convexity of the Lagrangian wrt (s, p).

(b) Let us consider the minimum problem

$$m = \min \{ F(u) : u \in C^1([0,2]),\ u(0) = 0 \}$$

It is possible to check that the min exists and the minimum point $u_1(x)$ is the unique solution to

$$\begin{cases} \ddot{u} = u - x^2 \\ u(0) = 0 \\ \dot{u}(2) = 0 \end{cases}$$

Since the BC $\dot{u}(2) = 0$ is not compatible with the given condition $\dot{u}(2) = 4$, we deduce that

- the minimum does not exist
- the infimum is m
- a minimizing sequence can be obtained by modifying $u_1(x)$ in a small left neighborhood of $x = 2$.

2. Discuss existence, uniqueness and regularity of solutions to equation

$$(\dot{u}^2 + e^{\dot{u}}) \cdot \ddot{u} = x^2 + e^u$$

with each of the following sets of boundary conditions:

(a) $u(-1) = u(1) = 0$,

(b) $u'(-1) = u'(1) = 0$.

(a) Let us consider the min. pbm.

$$\min \left\{ \int_{-1}^{1} \left(\frac{\dot{u}^4}{12} + e^{\dot{u}} + x^2 u + e^u \right) dx \; : \; u(-1) = u(1) = 0 \right\}$$

Any solution to this pbm. satisfies the given equ. with BCs (a). To this pbm. we can apply the standard direct method.

Existence The key point is that $L(x,s,p) \geq p^2 - |s| - A$ for a suitable A, so that the growth wrt p dominates the growth wrt s.

Uniqueness and regularity Standard due to the strict convexity of $L(x,s,p)$ wrt (s,p) and to the fact that $L_{ss} > 0$.

(b) The problem has $\boxed{\text{No solution}}$.
Indeed, any solution to the diff. equ. satisfies $\ddot{u}(x) > 0$ in $[-1,1]$, and hence $\dot{u}(1) > \dot{u}(-1)$.

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Alternative approach for point (b). Any solution to the equation with BCs $\dot{u}(\pm 1) = 0$ is a global minimizer for the functional with Lagrangian

$$L(x,s,p) = \frac{1}{12} p^4 + e^p - p + x^2 s + e^s$$

without BCs.

But this functional has no GM because the infimum is $-\infty$ (just use $u_n(x) \equiv -n$).

Question: what goes wrong in this case with the direct method?

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3. Let us consider, for every $\ell > 0$ and $\mu \in \mathbb{R}$, the problem

$$\min \left\{ \int_0^\ell (\dot{u}^2 + \sin u - u^2)\, dx : u \in C^1([0,\ell]),\ u(0) = 0,\ u(\ell) = \mu \right\}.$$

(a) Discuss the existence of the minimum in the case $\ell = \mu = 2018$.

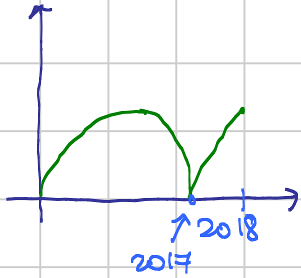
(b) Discuss the existence of the minimum in the case $\ell = 3$ and $\mu = 0$.

(c) Discuss the existence of the minimum in the case $\ell = 3$ and $\mu = 2018$.

(a) The minimum $\boxed{\text{does NOT exist and } \inf = -\infty}$

To this end, it is enough to observe that

- $\int_0^{2017} \dot{u}^2 - u^2$ is unbounded from below in the class with $u(0) = u(2017) = 0$
- The term with $\sin u$ is bounded
- We can define u in $[2017, 2018]$ s.t. $u(2017) = 0$ and $u(2018) = 2018$



(b) $\boxed{\text{The min DOES exist.}}$ Indeed, since $3 < \pi$, from the theory of quadratic functionals we know that

$$\int_0^3 (\dot{u}^2 - u^2 + \sin u)\, dx \geq \int_0^3 (\varepsilon_0 \dot{u}^2 + \sin u)\, dx$$

for a suitable $\varepsilon_0 > 0$. This is what we need in order to apply the direct method (any estimate on the functional yields an estimate on $\|\dot{u}\|_2$).

(c) $\boxed{\text{The min DOES exist}}$ Indeed, let us set $u = v + \frac{2018}{3} x$.

Thus we have to minimize

$$\int_0^3 \left(\dot{v} + \frac{2018}{3}\right)^2 - \left(v + \frac{2018}{3} x\right)^2 + \sin\left(v + \frac{2018}{3} x\right)$$

with BCs $v(0) = v(3) = 0$.

Now we can argue as in point (b) because $\int_0^3 \dot{v}^2 - v^2 \geq \varepsilon_0 \int_0 \dot{v}^2$ and all the other terms have at most linear growth.

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Alternative for (a): variable change as in point (c).

4. Let us set

$$m_\varepsilon := \inf\left\{\int_0^1 \left(\frac{\dot u^6}{\varepsilon^2} - \frac{\dot u^2}{\varepsilon^6} + \arctan u\right)\,dx : u \in C^1([0,1]),\ u(0)=0,\ u(1)=0\right\}.$$

(a) Determine for which positive values of ε it turns out that m_ε is finite.

(b) Determine for which positive values of ε it turns out that m_ε is actually a minimum.

(c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

(a) $\boxed{m_\varepsilon \text{ is finite for every } \varepsilon > 0}$ It is enough to observe that the Lagrangian is bounded from below.

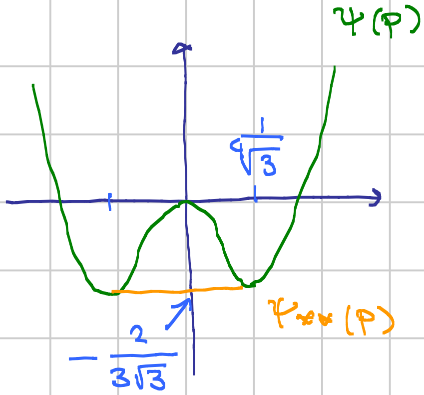
(b) $\boxed{m_\varepsilon \text{ IS NEVER a minimum}}$

Indeed, let us assume that u_0 is a minimizer. Due to BCs and Rolle's theorem, the exists $x_0 \in (0,1)$ s.t. $\dot u(x_0) = 0$. In that point $L_{pp}(x_0, u_0(x_0), \dot u_0(x_0)) < 0$, and hence (L) is violated.

(c) $\boxed{m_\varepsilon \sim -\frac{2}{3\sqrt{3}}\,\frac{1}{\varepsilon^8}}$ Indeed, setting $u(x) = \frac{1}{\varepsilon}v(x)$ we obtain that

$$\int_0^1 \frac{\dot u^6}{\varepsilon^2} - \frac{\dot u^2}{\varepsilon^6} + \arctan u = \frac{1}{\varepsilon^8}\underbrace{\int_0^1 \dot v^6 - \dot v^2 + \varepsilon^8 \arctan\left(\frac{v}{\varepsilon}\right)}_{G_\varepsilon(v)}$$

Now in a standard way it is possible to show that



$$G_\varepsilon(v) \xrightarrow{\Gamma} \int_0^1 \psi_{**}(\dot v)$$

whose minimum is $-\frac{2}{3\sqrt{3}}$.

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