1. Let us consider the functional

$$F(u) = \int_0^3 (\ddot{u}^2 + 3u)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u'(0) = 0$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

$$\delta F(u,v) = \int_0^3 (2\ddot{u}\ddot{v} + 3v) = 2[\ddot{u}\dot{v}]_0^3 - 2\int_0^3 \dddot{u}\,\dot{v} + 3\int_0^3 v$$

$$= 2[\ddot{u}\dot{v}]_0^3 - 2[\dddot{u}\,v]_0^3 + 2\int_0^3 u^{IV} v + 3\int_0^3 v$$

$$= 2\{\ddot{u}(3)\dot{v}(3) - \ddot{u}(0)\dot{v}(0) - \dddot{u}(3)v(3) + \dddot{u}(0)v(0)\} + \int_0^3 (2u^{IV}+3)v$$

(a) Admissible test functions satisfy $v(0) = \dot{v}(0) = 0$. We obtain

$$\begin{cases} u^{IV} = -\frac{3}{2} \\ u(0) = \dot{u}(0) = 0 & \rightsquigarrow \text{given BCs} \\ \ddot{u}(3) = \dddot{u}(3) = 0 & \rightsquigarrow \text{BCs originated by } \delta F \end{cases}$$

$\rightsquigarrow$ general solution is

$$u(x) = -\frac{1}{16}x^4 + ax^3 + bx^2 + cx + d$$

$\rightsquigarrow$ keeping BC into account the unique solution is

$$\boxed{u_0(x) = -\frac{1}{16}x^4 + \frac{3}{4}x^3 - \frac{27}{8}x^2}$$

It is the unique minimum point since

$$F(u_0 + v) = F(u_0) + \underbrace{\delta F(u_0, v)}_{=0} + \int_0^3 \ddot{v}^2 \geq F(u_0), \text{ with equality iff}$$

$\int \ddot{v}^2 = 0$, which implies $v \equiv 0$ because $v(0) = \dot{v}(0) = 0$.

(b) $\boxed{\operatorname{Inf} = -\infty}$ and a possible minimizing sequence is

$$u_n(x) := -nx$$

Remark See what happens when we apply the direct method to (a) or (b).

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$\ddot{u} = \sinh(x+u), \qquad u'(0) = 0, \quad u'(1) = 7.$$

Let us set $v(x) := u(x) - \frac{7}{2}x^2$, and let us observe that v solves

$$\ddot{v} = \ddot{u} - 7 = \sinh(x+u) - 7 = \sinh\left(v + \frac{7}{2}x^2 + x\right) - 7$$

with BCs

$$\dot{v}(0) = \dot{v}(1) = 0.$$

Let us consider the minimum problem

$$\min\left\{ \int_0^1 \left(\frac{1}{2}\dot{v}^2 + \cosh\left(v + \frac{7}{2}x^2 + x\right) - 7v\right)dx \ : \ u \in H^1((0,1)) \right\}$$

The given equ. in v is the (ELE) of this problem.
At this point the standard direct method works.

- Existence The key point is that $L(x,s,p) \geq \frac{1}{2}p^2 + s^2 - A$ for a suitable constant A. This provides integral bounds on $\dot{u}$ and u.

- Uniqueness Enough to observe that $L(x,s,p)$ is strictly convex in (s,p) for every $x \in [0,1]$, and therefore
  → every solution to the diff. pbm. is a minimum point
  → the minimum point is unique.

- Regularity Initial step + bootstrap as always.

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \{\tanh(\dot{u}^2) + \arctan(u^3 - u^2)\}\,dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Compute the infimum as a function of ℓ.

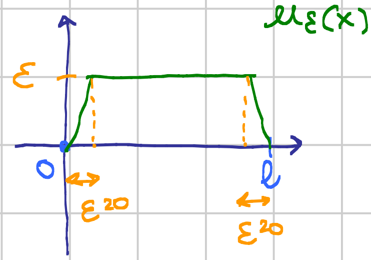
(a) u_0 is a WLM $\iff$ $\boxed{\ell < \pi}$ Indeed $\delta^2 F(u_0, v) = 2\int_0^\ell (\dot{v}^2 - v^2)$ and therefore

- if $\ell > \pi$ the function u_0 satisfies (L^+) but not (J)
- if $\ell < \pi$ the function u_0 satisfies all sufficient conditions: $(E) + (L^+) + (J)$
- if $\ell = \pi$ we observe that $F(u) < \int_0^\ell \dot{u}^2 - u^2 + u^3$ when u is small enough, and therefore

  $F(-\varepsilon \sin x) < 0$ when $\varepsilon > 0$ is small enough.

(b) u_0 is $\boxed{\text{never}}$ a SLM

Indeed, the functions $u_\varepsilon(x)$ described in the figure satisfy



$$F(u_\varepsilon) = -\ell\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+$$

(c) $\boxed{\inf = -\frac{\pi}{2}\ell \qquad \text{for every } \ell > 0}$

A possible minimizing sequence $u_n(x)$ is described in the figure.

Remark It could be interesting to interpret the result of point (c) in terms of relaxation.

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4. For every real number $\ell > 0$, let us set

$$I(\ell) := \inf \left\{ \int_0^\ell \left(\dot{u}^6 - \dot{u}^2 + u^2 - u^4\right) dx : u \in C^1([0,1]),\ u(0) = 0,\ u(\ell) = 0 \right\}.$$

(a) Determine for which positive values of ℓ it turns out that $I(\ell)$ is negative.

(b) Determine for which positive values of ℓ it turns out that $I(\ell)$ is finite.

(c) Compute the leading term of $I(\ell)$ as $\ell \to +\infty$.

(a) $\boxed{I(\ell) < 0 \text{ for every } \ell > 0}$ Indeed, $u_0(x) \equiv 0$ is not even a DLM because it does not satisfy (L)

(b) $\boxed{I(\ell) \text{ is finite for every } \ell > 0}$ Indeed, it turns out that

$$L(x, s, p) \geq \frac{1}{2} p^6 - s^4 - A$$

Since the growth wrt p dominates the growth wrt s, the functional

$$\int_0^\ell \frac{1}{2} \dot{u}^6 - u^4$$

is always bounded from below (keeping BCs into account, we can bound $\|u\|_\infty$ in terms of $\|\dot{u}\|_6$).

(c) We claim that $\boxed{I(\ell) \sim m\,\ell^{13}}$ where

$$\boxed{m := \min \left\{ \int_0^1 (\dot{w}^6 - w^4)\, dx : w(0) = w(1) = 0 \right\}}$$

Indeed, with the variable change $x = \ell y$ we obtain that

$$\int_0^\ell (\dot{u}^6 - \dot{u}^2 + u^2 - u^4)\, dx = \ell \int_0^1 (\dot{u}(\ell y)^6 - \dot{u}(\ell y)^2 + u(\ell y)^2 - u(\ell y)^4)\, dy$$

$$= \ell \int_0^1 \left(\frac{\dot{v}(y)^6}{\ell^6} - \frac{\dot{v}(y)^2}{\ell^2} + \frac{v(y)^2}{\ell^2} - \frac{v(y)^4}{\ell^4} \right) dy \qquad \text{where } v(y) = u(\ell y)$$

With the variable change $v(y) = \ell^3 w(y)$ we rewrite the latter as

$$\underbrace{\ell^{13} \int_0^1 \left(\dot{w}^6 - \frac{\dot{w}^2}{\ell^8} + \frac{w^2}{\ell^6} - w^4 \right)}$$

$$\downarrow \Gamma$$

$$\int_0^1 \dot{w}^6 - w^4$$

The verification of Γ-convergence is standard.

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