# Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 20 Luglio 2018

1. Let us consider the functional
$$F(u) = \int_0^3 \left(\ddot{u}^2 + 3u\right)\,dx.$$
   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u'(0) = 0$.
   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem
$$\ddot{u} = \sinh(x+u), \qquad u'(0) = 0, \quad u'(1) = 7.$$

3. Let us consider, for every $\ell > 0$, the problem
$$\inf\left\{\int_0^\ell \left\{\tanh(\dot{u}^2) + \arctan(u^3 - u^2)\right\}\,dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$
   (a) Determine for which values of $\ell$ the function $u_0(x) \equiv 0$ is a weak local minimum.
   (b) Determine for which values of $\ell$ the function $u_0(x) \equiv 0$ is a strong local minimum.
   (c) Compute the infimum as a function of $\ell$.

4. For every real number $\ell > 0$, let us set
$$I(\ell) := \inf\left\{\int_0^\ell \left(\dot{u}^6 - \dot{u}^2 + u^2 - u^4\right)\,dx : u \in C^1([0,\ell]),\ u(0) = 0,\ u(\ell) = 0\right\}.$$
   (a) Determine for which positive values of $\ell$ it turns out that $I(\ell)$ is negative.
   (b) Determine for which positive values of $\ell$ it turns out that $I(\ell)$ is finite.
   (c) Compute the leading term of $I(\ell)$ as $\ell \to +\infty$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.