1. Discuss the minimum problem

$$\min\left\{\int_0^1 \left(\dot{u}-x^2\right)^2 dx : u(0)=u(1)\right\}.$$

(ELE) $\quad (L_p)' = L_s \quad \Longleftrightarrow \quad (\dot{u}-x^2)'=0 \quad \Longleftrightarrow \quad \ddot{u} = 2x$

$\rightsquigarrow$ general solution is $\quad u(x) = \frac{x^3}{3} + a + bx$

BCs are $\boxed{u(0)=u(1)}$ and $\quad L_p|_{x=0} = L_p|_{x=1}$, namely

$\boxed{\dot{u}(0) = \dot{u}(1) - 1}$ $\quad \rightsquigarrow \quad$ $a = \frac{1}{3} + a + b$ $\quad \rightsquigarrow \quad b = -\frac{1}{3}$

$b = 1 + b - 1$

$\rightsquigarrow$ $\boxed{u(x) = \frac{x^3}{3} - \frac{x}{3} + a}$ is a solution to (ELE) + BCs for every $a \in \mathbb{R}$.

Due to the convexity of the Lagrangian in the pair (s,p), all these functions are minimum points.

<u>Alternative</u> Let $u_0(x)$ be the solution with $a=0$. Then for every admissible variation it turns out that

$$F(u_0+v) = F(u_0) + \delta F(u_0, v) + \int_0^1 \dot{v}^2 \geq F(u_0)$$

with equality iff $\dot{v} \equiv 0$ ...

<u>Question</u> Why uniqueness fails?

2. Discuss existence, uniqueness, and regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{u^3 - x^3}{1+\dot{u}^2}, \qquad u(0) = 1, \quad \dot{u}(1) = 3.$$

$$(1+\dot{u}^2)\,\ddot{u} = u^3 - x^3 \qquad \left(\dot{u} + \frac{1}{3}\dot{u}^3\right)' = u^3 - x^3$$

Let us consider the minimum pbm

$$\min\left\{ \int_0^1 \left(\frac{1}{2}\dot{u}^2 + \frac{1}{24}\dot{u}^4 - 27\dot{u} + \frac{1}{4}u^4 - x^3 u\right)dx \;:\; u(0) = 1 \right\}$$

The given equ. is (ELE) of this pbm, the DBC is the same, and the NBC is $L_p|_{x=1} = 0$, namely

$$\dot{u}(1) + \frac{1}{3}\dot{u}(1)^3 - 27 = 0, \text{ whose unique solution is } \dot{u}(1) = 3.$$

At this point the standard direct method works.

<u>Existence</u> The key point is that $L(x,s,p) \geq \frac{1}{4}p^2 - A|s| - B$ for suitable constants A, B.

Thus a bound on the functional provides a bound on $\|\dot{u}\|_{L^2}$.

<u>Uniqueness</u> Since $L(x,s,p)$ is strictly convex in (s,p), any solution to (ELE) + BCs is a minimizer, and the minimizer is unique.

<u>Regularity</u> Standard argument (initial step + bootstrap).

The key point is that $L_{pp} > 0$.

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left(-\cos(\dot u) + \cos(u)\right) dx : u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

(a) $\boxed{u_0 \text{ is WLM} \iff \ell \le \pi}$ Indeed $\delta^2 F(u_0, v) = \int_0^\ell \dot v^2 - v^2$ and therefore

- if $\ell < \pi$ then all sufficient conditions are satisfied, namely (E) + (L$^+$) + (J$^+$),
- if $\ell > \pi$ then u_0 satisfies (L$^+$) but not (J)
- if $\ell = \pi$ the situation is more delicate. Let us consider the ineq.

$$-\cos a + \cos b \ge \frac{1}{8}(a^2 - b^2) \qquad \forall (a,b) \in \text{neighborhood of } (0,0)$$

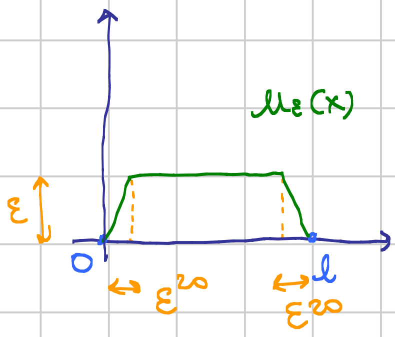
follows from example from sum to product formulae

It follows that

theory of quadratic functionals

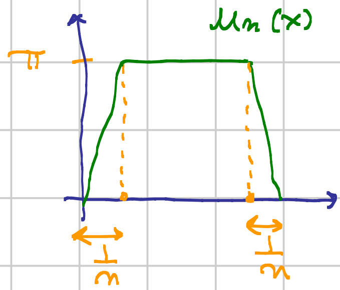
$$\int_0^\pi -\cos(\dot u) + \cos(u) \ge \frac{1}{8}\int_0^\pi \dot u^2 - u^2 \ge 0$$

(b) $\boxed{u_0 \text{ IS NEVER a SLM}}$



It is enough to observe that $F_\varepsilon(u_\varepsilon) \sim -\frac{1}{2}\varepsilon^2 \ell$, where u_ε is defined as in the figure

(c) $\boxed{\inf = -2\ell \quad \forall \ell > 0}$ Indeed



- $\inf \ge -2\ell$ is trivial
- -2ℓ can be approximated by the sequence $u_n(x)$ defined as in the figure.

Remark. Point (c) may be interpreted as a relaxation result.

4. Let us consider, for every $\varepsilon > 0$, the problem

$$m_\varepsilon = \min\left\{\int_0^1 (\sinh(\dot u^2) + u^6)\,dx : u(0) = u(1) = 0,\ \int_0^1 u^4\,dx = \varepsilon\right\}.$$

(a) Prove that the minimum exists for every $\varepsilon > 0$.

(b) Determine all real numbers α for which

$$\lim_{\varepsilon\to 0+} \frac{m_\varepsilon}{\varepsilon^\alpha} = 0.$$

(a) Standard direct method (the integral constraint is continuous wrt the usual notion of convergence)

(b) $\boxed{\alpha < \frac{1}{2}}$ Indeed $m_\varepsilon \sim c\sqrt{\varepsilon}$, where

$$c := \min\left\{\int_0^1 \dot v^2 : v(0) = v(1) = 0,\ \int_0^1 v^4\,dx = 1\right\}.$$

Setting $u = \sqrt[4]{\varepsilon}\, v$, the problem becomes

$$m_\varepsilon = \min\left\{\sqrt{\varepsilon}\ \underbrace{\int_0^1 \frac{\sinh(\sqrt{\varepsilon}\,\dot v^2)}{\sqrt{\varepsilon}} + \varepsilon v^6}_{G_\varepsilon(v)} : v(0) = v(1) = 0,\ \int_0^1 v^4 = 1\right\}$$

Now it turns out that $G_\varepsilon \xrightarrow{\Gamma} \int_0^1 \dot v^2$ with equicoerciveness.

The argument is quite standard:

→ liminf ineq.: use that

$$\frac{\sinh(\sqrt{\varepsilon}\,p^2)}{\sqrt{\varepsilon}} + \varepsilon s^6 \geq p^2$$

→ limsup ineq.: enough to do it when $u \in C^1$ + BCs + integral, in which case the constant recovery sequence works.

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