1. Let us consider the functional

$$F(u) = \int_0^\pi \left[\ddot{u}^2 + \dot{u}^2 + \sin x \cdot u\right]\, dx.$$

Discuss the minimum problem for $F(u)$ subject to each of the following boundary conditions:

(a) $u(0) = \pi$,

(b) $u'(0) = 2017$.

$$\delta F(u,v) = \int_0^\pi 2\ddot{u}\ddot{v} + 2\dot{u}\dot{v} + \sin x \cdot v$$

$$= 2\left[\ddot{u}\dot{v}\right]_0^\pi + 2\left[\dot{u}v\right]_0^\pi - \int_0^\pi 2\dddot{u}\dot{v} - \int_0^\pi 2\ddot{u}v + \int_0^\pi \sin x \cdot v$$

$$= 2\left[\ddot{u}\dot{v} + \dot{u}v - \dddot{u}v\right]_0^\pi + \int_0^\pi (2u^{IV} - 2\ddot{u} + \sin x)v$$

(a) In this case admissible variations satisfy $v(0) = 0$, and hence

$$\begin{cases} u^{IV} - \ddot{u} = -\frac{1}{2}\sin x \\ u(0) = \pi \\ \ddot{u}(0) = 0 \\ \ddot{u}(\pi) = 0 \\ \dot{u}(\pi) = \dddot{u}(\pi) \end{cases}$$

$\rightsquigarrow$ general solution is

$$u(x) = -\frac{1}{4}\sin x + a + bx + c\cosh x + d\sinh x$$

Keeping BCs into account, we obtain that $a = \pi$, $b = -\frac{1}{2}$, $c = d = 0$.
It follows that

$$\boxed{u_0(x) = -\frac{1}{4}\sin x + \pi - \frac{1}{2}x}$$

is the unique solution to (ELE), and hence (due to the convexity of the Lagrangian) the unique solution to the minimum pbm.

(In alternative: $F(u+v) \geq F(u) + \int_0^\pi \ddot{v}^2 + \int_0^\pi \dot{v}^2$ for every admissible v, and therefore ...)

(b) The min. $\boxed{\text{does NOT exist and } \inf = -\infty}$ and a min. seq. is

$$u_m(x) = 2017x - m$$

Question: what goes wrong with the direct method?

2. Let us consider the boundary value problem

$$\ddot{u} = \arctan x \cdot \arctan u, \qquad u(0) = \pi, \quad u(\pi) = 0.$$

(a) Discuss existence, uniqueness, and regularity of the solution.

(b) Discuss the monotonicity of the solution.

(a) Let $g(x) := \int_0^x \arctan s \, ds$. Let us consider the min. pbm.

$$\min \left\{ \int_0^\pi \frac{1}{2} \dot{u}^2 + \arctan x \cdot g(u) \; : \; u(0) = \pi, \; u(\pi) = 0 \right\}$$

The given eqn. with DBC is (ELE) of this pbm. The standard direct method works.

- Existence: the key point is that $\arctan x \geq 0$ in $[0, \pi]$ and $g(u) \geq 0$ always. Thus any bound on the functional yields a bound on $\|\dot{u}\|_{L^2}$.
- Uniqueness: follows from the convexity of the Lagrangian wrt to (s, p), strict wrt p.
- Regularity: standard argument (initial step + bootstrap)

(b) A standard truncation argument shows that

$$u(x) \geq 0 \qquad \forall x \in [0, \pi]$$

(can be refined to $u(x) > 0$ for every $x \in [0, \pi)$).

As a consequence

- $\dot{u}(\pi) \leq 0$
- $\ddot{u}(x) \geq 0$ for every $x \in [0, \pi]$ (from the equation)

Thus from the monotonicity of $\dot{u}(x)$ we deduce that

$$\dot{u}(x) \leq 0 \qquad \forall x \in [0, 1]$$

and therefore $\boxed{u(x) \text{ is nonincreasing}}$ (and actually decreasing)

More refined argument: assume that $u(x_0) = 0$ for some $x_0 \in (0, \pi)$

Then also $\dot{u}(x_0) = 0$, and therefore ...

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell [\arctan(\dot u^2) + \sin(u^2)]\, dx : u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

(a) $\boxed{u_0 \text{ is WLM } \forall \ell > 0}$ Indeed the second variation is

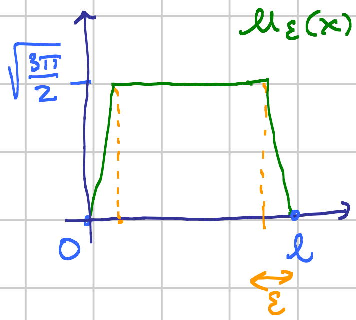
$$\delta^2 F(u_0, v) = 2\int_0^\ell (\dot v^2 + v^2)$$

and therefore u_0 satisfies all sufficient conditions $(E)+(L^+)+(J^+)$.

(b) $\boxed{u_0 \text{ is SLM } \forall \ell > 0}$ Indeed $F(u) \geq 0$ for every $u : [0,\ell] \to \mathbb{R}$ with $\underbrace{|u(x)| \leq 1 \quad \forall x \in [0,1]}_{C^0 \text{ neighborhood of } u_0}$

(c) $\boxed{\inf = -\ell \qquad \forall \ell > 0}$ Indeed the lower bound $F(u) \geq -\ell$ is trivial, and at this point it is enough to define $u_\varepsilon(x)$ as in the figure and observe that

$$F(u_\varepsilon) \to -\ell \quad \text{as } \varepsilon \to 0^+.$$



<u>Remark</u> It is interesting to interpret the result of point (c) in terms of relaxation (note that 0 is the convexification of $\arctan(p^2)$).

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4. For every positive integer n, let us set

$$M_n = \inf\left\{\int_0^1 \left(2017\dot{u}^2 + nu^6 - n^2u^2\right)\,dx : u(0) = u(1) = 0\right\}.$$

(a) Determine for which values of n the infimum is actually a minimum.

(b) Determine for which values of n the infimum is negative.

(c) Determine the leading term of M_n as $n \to +\infty$.

(a) $\boxed{\inf = \min \quad \forall n \geq 1}$ Indeed for every $n \geq 1$ there exists $A_n \geq 0$ such that

$$ns^6 - n^2 s^2 \geq -A_n \qquad \forall s \in \mathbb{R}$$

This is enough to apply the standard direct method (an estimate on the functional yields an estimate on $\|\dot{u}\|_{L^2}$).

(b) $\boxed{\inf = \min < 0 \iff n > \pi\sqrt{2017}}$ Indeed

- if $n \leq \pi\sqrt{2017}$ it turns out that

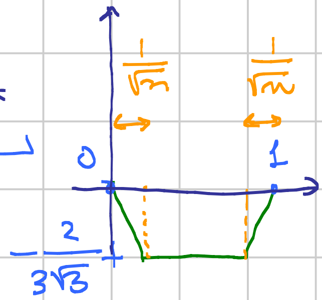
$$F_n(u) \geq \int_0^1 2017\dot{u}^2 - n^2u^2 \geq 0$$

the quadratic functional satisfies (L^+) and (J^+)

- if $n > \pi\sqrt{2017}$ the function $u_0(x) \equiv 0$ satisfies (L^+) but not (J) (the second variation is the quadratic functional above) and therefore $u_0(x) \equiv 0$ is not even a DLM.

(c) Setting $u = \sqrt[4]{n}\,v$ we find that

$$\int_0^1 2017\dot{u}^2 + nv^6 - n^2v^2 = n^2\sqrt{n}\underbrace{\int_0^1 \left(\frac{\dot{v}^2}{n^2} + v^6 - v^2\right)dx}_{G_n(v)}$$



and it is easy to check that

$$\min\{G_n(v) : v(0) = v(1) = 0\} \to \min\{s^6 - s^2 : s \in \mathbb{R}\} = -\frac{2}{3\sqrt{3}}$$

It follows that $\boxed{M_n \sim -\frac{2}{3\sqrt{3}}\,n^2\sqrt{n}}$ [Also $G_n(v) \xrightarrow{\Gamma} \int_0^1 (v^6 - v^2)$ without BCs]

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