1. Discuss the minimum problem

$$\min\left\{\int_0^1 \left[(\dot u + x)^2 + (u+x)^2\right]\,dx : u'(1) = \alpha\right\}$$

depending on the real parameter α.

(ELE) $(L_p)' = L_s$ $\qquad (\dot u + x)' = u + x$, $\quad \ddot u = u + x - 1$

$$\Rightarrow u(x) = a\cosh x + b\sinh x - x + 1$$

NBC : $L_p = 0$ $\qquad \dot u + x = 0$ $\qquad$ in $x = 0 \rightsquigarrow \dot u(0) = 0$

in $x = 1 \rightsquigarrow \dot u(1) = -1$

Let us consider the minimum pbm. without any BC. The solution is the solution to (ELE) with NBC

$$\begin{cases} \dot u(0) = 0 \\ \dot u(1) = -1 \end{cases} \iff \begin{cases} b - 1 = 0 \\ a\sinh 1 + b\cosh 1 - 1 = -1 \end{cases} \rightsquigarrow \text{unique solution } u_0(x)$$

and $u_0(x)$ is GM because of strict convexity in (p,s) of the Lagrangian.

Thus we distinguish two cases:

$\boxed{\alpha = -1}$ The pbm. admits a solution, and $u_0(x)$ is the unique min. point.

$\boxed{\alpha \neq -1}$ The min. does not exist and the infimum is $F(u_0)$, with u_0 as before (it is enough to modify $u_0(x)$ in a neighborhood of $x = 1$).

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2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{u^3}{u^2+1} \cdot |\cos x|, \qquad u'(0) = 1, \quad u(1) = 0.$$

Let us observe that $\frac{s^3}{s^2+1}$ is an increasing function and hence its antiderivative

$$\int_0^s \frac{t^3+t-t}{t^2+1}\, dt = \frac{1}{2}s^2 - \frac{1}{2}\log(1+s^2)$$

is a CONVEX and nonnegative function.

The GIVEN BVP is the ELE of

$$\min\left\{ \int_0^1 \left\{ \frac{1}{2}(\dot{u}-1)^2 + \left[\frac{1}{2}u^2 - \frac{1}{2}\log(1+u^2)\right] \cdot |\cos x| \right\} dx \ : \ u(1) = 0 \right\}$$

note that the NBC in $x=0$ is $L_p = 0$, namely $\dot{u}(0) = 1$.

Since $g(s)$ is bounded from below, any bound on $F(u)$ yields a bound on $\|\dot{u}\|_{L^2}$, and also a bound on $\|u\|_{L^\infty}$ because of the DBC.

Standard direct method $\Rightarrow$ existence

Strict convexity of $L(x,s,p)$ w.r.t. (s,p) $\Rightarrow$ uniqueness

$L_{pp}(x,s,p) > 0$ $\Rightarrow$ regularity via bootstrap, up to C^∞ (note that $|\cos x|$ is of class C^∞ in $[0,1]$.

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3. Let us consider, for every $\ell > 0$, the problem

$$\min\left\{\int_0^\ell \left[\sinh(\dot{u}^2) - \sin(u^2) + u^4\right] dx : u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the minimum exists.

(b) Determine for which values of ℓ the minimum (exists and) is negative.

(c) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

a) The Lagrangian is of the form $\varphi(p) + \psi(s)$, with $\varphi(p)$ convex and superquadratic, and $\psi(s)$ bounded from below.
Standard direct method $\Rightarrow$ existence and regularity.

Conclusion: $\boxed{\text{min exists for every } \ell > 0}$

(b) The quadratic functional $Q_\ell(v) := \int_0^\ell (\dot{v}^2 - v^2)\,dx$ is $\geq 0 \iff \ell \leq \sqrt{\pi}$.
We distinguish two cases.

$\boxed{\ell \leq \sqrt{\pi}}$ We observe that $F(u) \geq Q_\ell(v) + \int_0^\ell u^4 \geq 0$
with equality $\iff u \equiv 0$.

$\boxed{\ell > \sqrt{\pi}}$ min < 0. Indeed, if min $= 0$ the function $u_0(x) \equiv 0$ is a GM, and hence the second variation should be ≥ 0.
But the second variation is $Q_\ell(v)$.

Conclusion: $\boxed{\min < 0 \iff \ell > \sqrt{\pi}}$

(c) Again we distinguish two cases.

$\boxed{\ell > \sqrt{\pi}}$ $u_0(x) \equiv 0$ is not WLM $\Rightarrow$ it is not SLM

$\boxed{\ell \leq \sqrt{\pi}}$ We have just proved that $u_0(x) \equiv 0$ is GM.

Conclusion: $\boxed{u_0(x) \equiv 0 \text{ is SLM} \iff u_0(x) \text{ is GM} \iff \ell \leq \sqrt{\pi}}$

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$$m_\varepsilon = \inf\left\{\int_0^1 (\varepsilon \dot{u}^2 + \arctan \dot{u} + \arctan u)\, dx : u(0) = 3,\ u(1) = 2017\right\},$$

where ε is a positive real parameter.

(a) Determine if there exist values of ε for which the infimum is actually a minimum.

(b) Determine if there exist values of ε for which the infimum is not a minimum.

(c) Compute the limit of m_ε as $\varepsilon \to 0^+$.

The Lagrangian is of the form $\varphi_\varepsilon(p) + \psi(s)$

(a) If ε is large enough $\varphi_\varepsilon(p)$ is convex and $\psi(s)$ is bounded from below.
Standard direct method $\Rightarrow$ existence and regularity

(b) If ε is small enough, then the inf is not a min.
Indeed, assume that $u_0(x)$ is a GM. Due to DBC and Rolle's thm, it follows that $\dot{u}(x_0) = 2014$ for some $x_0 \in (0,1)$, and hence

$$L_{pp}(x_0, u(x_0), \dot{u}(x_0)) = \varphi_\varepsilon''(2014) < 0.$$

This means that (L) fails, and therefore $u_0(x)$ is not even a DLM

(c) We claim that $F_\varepsilon(u) \xrightarrow{\Gamma} -\frac{\pi}{2} + \int_0^1 \arctan u\, dx =: F(u)$

where $F(u)$ is thought WITHOUT DBC. This requires 2 steps

- Γ-liminf is almost trivial because $F_\varepsilon(u) \geq F(u)$ for every admissible u and ε.
- Γ-limsup follows from two facts
  - * $\Gamma\text{-limsup} \leq \int_0^1 (\arctan \dot{u} + \arctan u)\, dx =: G(u)$ if $u \in C^1 + \text{DBC}$ (just use the constant recovery sequence)
  - * the relaxation of $G(u)$ is $F(u)$ (standard convexification + loss of DBC).

At this point we can conclude that $\boxed{m_\varepsilon \to \min\{F(u)\} = -\pi}$

Indeed $m_\varepsilon > \pi$ for every $\varepsilon > 0$, and then it is enough to take recovery sequences for large negative constant functions.