1. Let us consider the two functionals

$$F_1(u) = \int_0^2 \{(\dot u - 2x)^2 + (u - x^2)^2\}\, dx, \qquad F_2(u) = \int_0^2 \{(\dot u - 2x) + (u - x^2)^2\}\, dx.$$

(a) Discuss the minimum problem for $F_1(u)$ subject to the boundary condition $u(0) = u(2)$.

(b) Discuss the minimum problem for $F_2(u)$ subject to the boundary condition $u(0) = u(2)$.

(a) (ELE) $\rightsquigarrow$ $[\cancel{2}(\dot u - 2x)]' = \cancel{2}(u - x^2)$ $\rightsquigarrow$ $\ddot u = u + 2 - x^2$

(BC) $u(0) = u(2)$ and $\dot u - 2x|_{x=0} = \dot u - 2x|_{x=2}$

$\rightsquigarrow$ $\dot u(0) = \dot u(2) - 4$

$$\begin{cases} \ddot u = u + 2 - x^2 \\ u(0) = u(2) \\ \dot u(0) = \dot u(2) - 4 \end{cases}$$

$\rightsquigarrow$ general solution $u(x) = a\cosh x + b\sinh x + x^2$

$a = a\cosh 2 + b\sinh 2 + 4$

$b = a\sinh 2 + b\cosh 2$

The system has a unique solution $u_0(x)$, which is the $\boxed{\text{unique minimum point}}$ due to the strict convexity of $L(x,s,p)$ in the pair (s,p)

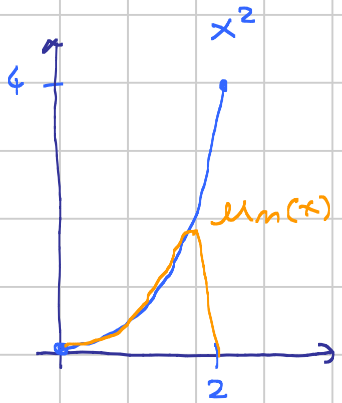
(b) $$F_2(u) = \int_0^2 \dot u - 2\int_0^2 x + \int_0^2 (u - x^2)^2$$

$$= \underbrace{u(2) - u(0)}_{=0} - 4 + \underbrace{\int_0^2 (u - x^2)^2}_{\geq 0} \geq -4$$

We claim that $\boxed{\inf\{F_2(u) : u(0) = u(2)\} = -4}$, but it is $\boxed{\text{not a minimum}}$.

Indeed

- the only function for which $F_2(u) = -4$ is $u(x) = x^2$, and this does not satisfy the BCs



- it is possible to approximate x^2 in L^2 norm with functions $u_n(x)$ of class C^1 (and also C^∞) with $u_n(0) = u_n(2) = 0$.

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2. Let us consider the boundary value problem

$$\ddot{u} = \frac{x^7 u^9}{\dot{u}^8 + 10}, \qquad u'(0) = 11, \quad u(11) = 0.$$

Discuss existence, uniqueness and regularity of the solution.

$$\dot{u}^8 \ddot{u} + 10\ddot{u} = x^7 u^9 \quad \rightsquigarrow \quad \Big(\underbrace{\frac{\dot{u}^9}{9} + 10\dot{u}}_{L_p}\Big)' = \underbrace{x^7 u^9}_{L_s}$$

$$L(x,s,p) = \frac{1}{90} p^{10} + 5p^2 - \lambda p + \frac{1}{10} x^7 s^{10} \qquad \text{with} \quad \lambda = \frac{11^9}{9} + 110$$

Given equation is (ELE) and $\dot{u}(0) = 11$ is $L_p(x,u,\dot{u})\big|_{x=10} = 0.$

↑ to be verified

Therefore, we can consider the minimum problem

$$\min\left\{ \int_0^{11} \left(\frac{1}{90}\dot{u}^{10} + 5\dot{u}^2 - \lambda \dot{u} + \frac{1}{10} x^7 u^{10} \right) dx \; : \; u(11) = 0 \right\}$$

- Existence follows from the direct method in H^1 or $W^{1,10}$.

  The key point is that

$$L(x,S,p) \geq p^2 - A$$

  and therefore $F(u) \leq M \implies \|\dot{u}\|_{L^2} \leq M + 11A$

  $+ \text{BC} \implies$ compactness

- Uniqueness follows from the strict convexity of $L(x,s,p)$ in (s,p).

- Regularity follows in a standard way (initial step + bootstrap).

  The key point is that

$$L_{pp}(x,s,p) > 0 \qquad (\text{thanks to the term } 5p^2)$$

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Note: when deriving (ELE) in integral form for the weak solution, we need to know that $\dot{u}$ has enough summability (needed when taking the derivative of the parametric integral).

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3. Let us consider, for every $\lambda > 0$, the problem

$$m_\lambda := \inf\left\{\int_0^4 (\dot{u}^2 - \lambda u \arctan u + \sin^4 u)\, dx : u \in C^1([0,4]),\ u(0) = u(4) = 0\right\}.$$

(a) Determine for which values of λ the infimum is actually a minimum.
(b) Determine for which values of λ the infimum is negative.
(c) Determine the leading term of m_λ as $\lambda \to +\infty$.

(a) The infimum is $\boxed{\text{always a minimum}}$

Indeed the Lagrangian is of the form $p^2 + g(x,s)$ with
$$g(x,s) \geq -A|s| - B$$
(linear growth in s vs quadratic growth in p)
Therefore the standard direct method works.

(b) The min is negative $\boxed{\iff \lambda > \frac{\pi^2}{16}}$. Indeed

- if $\lambda \leq \frac{\pi^2}{16}$ it turns out that $F(u) \geq \underbrace{\int_0^4 \dot{u}^2 - \lambda u^2}_{\text{non-negative quadratic functional}} \geq 0$

- if $\lambda > \frac{\pi^2}{16}$ the function $u_0(x) \equiv 0$ satisfies (L^+) but not (J)

$\left(\delta^2 F(u_0, v) = 2\int_0^4 (\dot{v}^2 - \lambda v^2)\, dx \right)$ and therefore $u_0(x)$ is not a min. point.

(c) $\boxed{m_\lambda \sim \lambda^2 \min\left\{ \int_0^4 \left(\dot{v}^2 - \frac{\pi}{2}|v|\right) dx : v(0) = v(4) = 0 \right\}}$ (the coeff. can be computed by indirect method)

please note the abs. value

Indeed

- Setting $u = \lambda v$ we obtain that $m_\lambda \geq \lambda^2 \int_0^4 \left(\dot{v}^2 - \frac{\pi}{2}|v|\right) \geq \lambda^4 \min$

- Let $u_0(x)$ be the minimizer of the coefficient. then

$$m_\lambda \leq F(\lambda u_0) = \lambda^2 \underbrace{\int_0^4 \Big(\dot{u}_0^2 - u_0 \underbrace{\arctan(\lambda u_0)}_{\downarrow \pm\frac{\pi}{2}} + \underbrace{\tfrac{1}{\lambda^2} \sin^4(\lambda u_0)}_{\downarrow 0}\Big)\, dx}_{\downarrow \int_0^4 \dot{u}_0^2 - \frac{\pi}{2}|u_0|}$$

$\underline{\quad}\ \mathrm{o}\ \underline{\quad}\ \mathrm{o}\ \underline{\quad}$

4. Let us consider, for every value of the real parameter a, the minimum problem

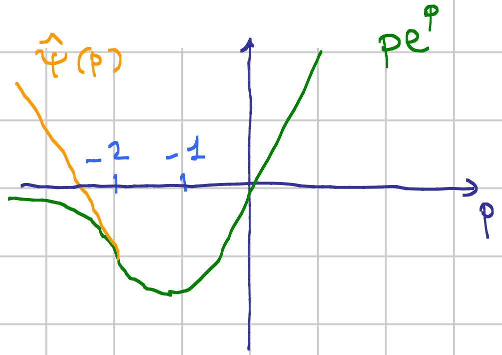
$$\min\left\{\int_0^1 u'(x)\cdot e^{u'(x)}\,dx : u\in C^1([0,1]),\ u(0)=0,\ u(1)=a\right\}.$$

(a) Determine for which values of a the function $u(x)=ax$ is a weak local minimum.
(b) Determine for which values of a the function $u(x)=ax$ is a strong local minimum.
(c) Determine for which values of a the minimum exists.

$\psi(p) = pe^p$

$\dot\psi(p) = 0 \iff p = -1$

$\ddot\psi(p) = 0 \iff p = -2$



(a) $\boxed{\text{WLM} \iff a > -2}$ Indeed

- if $a > -2$ then $u(x) = ax$ is the GM of $\int_0^1 \hat\psi(\dot u)\,dx$, and therefore it minimizes $\int_0^1 \psi(\dot u)\,dx$ in the C^1 neighborhood $\dot u > -2$
- if $a < 2$, then $u(x) = ax$ does not satisfy (L) because $L_{pp} = \ddot\psi(a)$
- if $a = -2$ it is enough to observe tha each neighborhood of -2 contains two points $a_\varepsilon < -2 < b_\varepsilon$ such that
  $\lambda_\varepsilon a_\varepsilon + (1-\lambda_\varepsilon) b_\varepsilon = -2$ and $\lambda_\varepsilon \dot\psi(a_\varepsilon) + (1-\lambda_\varepsilon)\dot\psi(b_\varepsilon) < \dot\psi(-2)$,
  and therefore a combination of slopes a_ε and b_ε is better than a slope $\equiv -2$.

(b) $\boxed{\text{SLM} \iff a \geq -1}$ Indeed

- if $a \geq -1$ the function $u(x) = ax$ is GM of $\int_0^1 \psi_{**}(\dot u)$, and a fortiori a GM of $\int_0^1 \psi(\dot u)$
- if $a < 1$ the function $u(x) = ax$ does not satisfy (W)

(c) $\boxed{\text{min exists} \iff a \geq -1}$ Indeed

- if $a \geq -1$ we already know that $u(x) = ax$ is GM of $\int_0^1 \psi_{**}(\dot u)$
- if $a < 1$ the inf is the min of $\int_0^1 \psi_{**}(\dot u) = \dot\psi(-1)$, which can not be achieved because $\dot u(x) < -1$ in some interval.

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