CdV 2018 - Exam paper 1 - 22.01.2018

Note Title 21/01/2018

1. Let us consider the functional

$$F(u) = \int_0^{\pi} \left(\dot u^2 - u\cos x\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi) = u(0)$.

(b) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi) = 2u(0)$.

$$\delta F(u,v) = \int_0^{\pi} (2\dot u \dot v - v\cos x)\, dx$$

(a) In a usual way we obtain (ELE) with (PBC)

$$\begin{cases} \ddot u = -\frac{1}{2}\cos x & \rightsquigarrow \quad u(x) = a + bx + \frac{1}{2}\cos x \\ u(0) = u(\pi) & \cancel{a} + \frac{1}{2} = \cancel{a} + b\pi - \frac{1}{2} \quad \rightsquigarrow \quad b = \frac{1}{\pi} \\ \dot u(0) = \dot u(\pi) & b = b \end{cases}$$

Therefore (ELE) + (PBC) has infinitely many solutions of the form

$$\boxed{u(x) = a + \frac{1}{\pi}x + \frac{1}{2}\cos x \qquad a \in \mathbb{R}}$$

These are all minimum points due to the convexity in (s,p) of the Lagrangian (which is NOT strictly convex, and hence ∞ many min. points is OK).

(b) $$\delta F(u,v) = \int_0^{\pi} (-2\ddot u - \cos x)v\, dx + 2\left[\dot u v\right]_0^{\pi}$$

$$= 2\left(\dot u(\pi)v(\pi) - \dot u(0)v(0)\right)$$

$$\underset{v(\pi)=2v(0)}{=} 2v(0)\left(2\dot u(\pi) - \dot u(0)\right)$$

In the usual way we obtain (ELE) with BC: $u(\pi) = 2u(0)$

$2\dot u(\pi) = \dot u(0)$

$$a + b\pi - \frac{1}{2} = 2a + 1$$

$$b = 2b \qquad \rightsquigarrow \quad b = 0 \quad \rightsquigarrow \quad a = -\frac{3}{2}$$

$\Rightarrow$ $\boxed{u(x) = -\frac{3}{2} + \frac{1}{2}\cos x \quad \text{is the unique min. point.}}$ (again by convexity)

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{\sinh u - 3}{3 + \cosh \dot{u}}, \qquad u(0) = u(2017) = 3.$$

(a) Discuss existence, uniqueness, regularity of the solution.

(b) Prove that the solution satisfies $1 < u(x) \le 3$ for every $x \in [0, 2017]$.

$(3 + \cosh \dot{u})\,\ddot{u} = \sinh u - 3 \quad ; \quad (3\dot{u} + \sinh \dot{u})' = \sinh u - 3$

$$F(u) = \int_0^{2017} \left(\frac{3}{2}\dot{u}^2 + \cosh \dot{u} + \cosh u - 3u \right) dx$$

Let us consider $\min \{ F(u) : u \in H^1((0,2017)),\ u(0) = u(2017) = 3 \}$

(a) Classical application of the direct method.

(CPT) Just observe that $F(u) \ge \int_0^{2017} \dot{u}^2 - A$ for suitable A.

Therefore $F(u) \le M_1 \Rightarrow \|\dot{u}\|_{L^2} \le M_2$ + DBC...

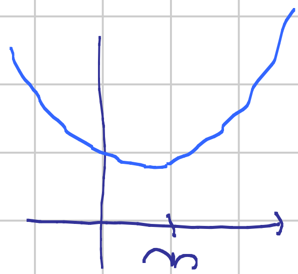
(LSC) Classical

(Regularity) Classical because $3 + \cosh u \ge 3$

(Uniqueness) Strict convexity in (s,p) of the Lagrangian.

(b) The function $\cosh u - 3u$ is convex with minimum point $m \in (1,3)$ (where $\sinh m = 3$).

Therefore, a classical truncation argument shows that $m \le u(x) \le 3$ for every $x \in [0, 2017]$.



Alternative ODE argument. Assume that $u(x) > 3$ somewhere. Then $\max u > 3$, and it is attained in some $x_0 \in (0, 2017)$. Now from the equation we deduce that $\ddot{u}(x_0) > 0$, which is absurd.

An analogous argument works with the minimum point.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell (\dot u^2 + \arctan u - u^2)\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the infimum is negative (possibly $-\infty$).

(b) Determine for which values of ℓ the infimum is actually a minimum.

(a) The infimum is negative for every $\ell > 0$. If not, $u_0(x) \equiv 0$ is a minimum point, but it does not satisfy ELE.

(b) We distinguish 3 cases.

- If $\ell > \pi$, then $\inf\left\{\int_0^\ell \dot u^2 - u^2 : u(0) = u(\ell) = 0\right\} = -\infty$, and hence also this infimum is $-\infty$.

- If $\ell < \pi$, then the quadratic functional $\int_0^\ell \dot u^2 - u^2$ is positive, and hence there exists $\varepsilon_0 > 0$ s.t.

$$\int_0^\ell \dot u^2 - u^2 \geq \varepsilon_0 \int_0^\ell \dot u^2$$

This is what we need in order to apply the direct method

- If $\ell = \pi$, then the quadratic functional is nonnegative, and hence

$$\int_0^\pi \dot u^2 - u^2 + \arctan u > -\frac{\pi}{2} \cdot \pi$$

On the other hand, the sequence $u_n(x) = -n \sin x$ approaches the infimum.

In conclusion

$\ell > \pi$	$\rightsquigarrow$	$\inf = -\infty$, NO min
$\ell = \pi$	$\rightsquigarrow$	$\inf = -\frac{\pi^2}{2}$, NO min
$\ell < \pi$	$\rightsquigarrow$	$\inf = \min \in (-\infty, 0)$

4. Let us set

$$m_\varepsilon = \min\left\{\int_0^1 \left(\varepsilon\ddot{u}^2 + \cos\dot{u} + \cos u\right)dx : u \in C^2([0,1]),\ u(0) = u'(0) = 1\right\}.$$

(a) Prove that m_ε is well-defined (namely the minimum exists) for every positive ε.

(b) Compute the limit of m_ε as $\varepsilon \to 0^+$.

(a) Standard direct method in $H^2((0,1))$. The parts in $\dot{u}$ and u are bounded from below $\rightsquigarrow$ bound on functional provides bound on u an $\ddot{u}$ $\rightsquigarrow$ CPT

(b) $\boxed{m_\varepsilon \to -2 \text{ as } \varepsilon \to 0^+}$ Let F_ε denote the functional.

It is easy to see that $\Gamma\text{-}\limsup F_\varepsilon(u) \leq \int_0^1 \cos\dot{u} + \cos u$ if $u \in C^2$ satisfies the BCs.
This can be proved using constant recovery sequences.
Since the Γ-limsup is LSC, it is $\leq$ than the relaxation of the RHS, which is

$$\int_0^1 (-1 + \cos u) =: F_0(u)$$

Since $F_0(u) \leq F_\varepsilon(u)$ for every $u \in L^2$, we have proved that

$$\Gamma\text{-}\lim_{\varepsilon \to 0^+} F_\varepsilon(u) = F_0(u)$$

and therefore

$$\liminf m_\varepsilon \leq \inf F_0 = -2$$

The other inequality is trivial because $F_\varepsilon(u) \geq F_0(u)$.

Alternative brute force approach: consider almost minimizing sequences as in the figure