Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 08 Giugno 2018

1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\dot{u}^2 + xu + u^2\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_{-1}^{1} u(x)\, dx = 7$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u(0) = 7$.

2. Let us consider, for any value of the real parameter a, the boundary value problem

$$\ddot{u} = (x+7)(u+7), \qquad u(-7) = u(7) = a.$$

   (a) Discuss existence, uniqueness and regularity of the solution.

   (b) Determine the values of a for which the solution is convex.

3. Let us set, for every $\ell > 0$,

$$I(\ell) := \inf\left\{\int_0^{\ell} \left(\cos x \cdot \dot{u}^2 + x^2 \cdot \cos u\right)\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

   (a) Determine the value of $I(2)$.

   (b) Determine whether $I(3/2)$ is actually a minimum.

   (c) Determine the value of $I(1)$.

4. Let us set

$$m_\varepsilon := \inf\left\{\int_0^1 \left(\varepsilon \dot{u}^6 - \dot{u}^2 + \sin u\right)\, dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = 0\right\}.$$

   (a) Determine for which positive values of ε it turns out that m_ε is finite.

   (b) Determine for which positive values of ε it turns out that m_ε is actually a minimum.

   (c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.