Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 19 February 2018

1. Let us consider the functional

$$F(u) = \int_0^1 \left\{ \left(\dot{u} - x^2\right)^2 + u \right\} dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = u(0) + 3$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{7 + u}{7 + \sin \dot{u}}, \qquad \dot{u}(0) = 7, \quad u(7) = 0.$$

   Discuss existence, uniqueness, regularity of the solution.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left(\arctan(\dot{u}^2) - \sin(u^2)\right) dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Compute the infimum as a function of ℓ.

4. Let us set

$$m_\varepsilon = \min \left\{ \int_0^1 \left(\dot{u}^2 + \dot{u}^6 + \varepsilon \sin u\right) dx : u \in C^1([0, 1]),\ u(0) = 0,\ u(1) = \varepsilon \right\}.$$

   (a) Prove that m_ε is well-defined (namely the minimum exists) for every positive ε.

   (b) Determine for which positive values of ε all minimum points are convex.

   (c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.