Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 22 January 2018

1. Let us consider the functional

$$F(u) = \int_0^\pi \left(\dot{u}^2 - u\cos x\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi) = u(0)$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi) = 2u(0)$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{\sinh u - 3}{3 + \cosh \dot{u}}, \qquad u(0) = u(2017) = 3.$$

   (a) Discuss existence, uniqueness, regularity of the solution.

   (b) Prove that the solution satisfies $1 < u(x) \leq 3$ for every $x \in [0, 2017]$.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left(\dot{u}^2 + \arctan u - u^2\right)\,dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

   (a) Determine for which values of ℓ the infimum is negative (possibly $-\infty$).

   (b) Determine for which values of ℓ the infimum is actually a minimum.

4. Let us set

$$m_\varepsilon = \min\left\{\int_0^1 \left(\varepsilon\ddot{u}^2 + \cos\dot{u} + \cos u\right)\,dx : u \in C^2([0,1]),\ u(0) = u'(0) = 1\right\}.$$

   (a) Prove that m_ε is well-defined (namely the minimum exists) for every positive ε.

   (b) Compute the limit of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.