BY GIMUSI 11/10/17
REV.01 12/10/17

Limiti 1

Argomenti: limiti di funzioni di più variabili **Difficoltà:** ⋆⋆⋆

Prerequisiti: tecniche per il calcolo di limiti in un punto per funzioni di più variabili

In ogni riga è assegnata una funzione, di cui si chiede di calcolare liminf e limsup per $(x, y) \to (0, 0)$. Nelle varie colonne, la funzione si intende definita nel suo "naturale dominio" intersecato l'insieme definito dalle relazioni indicate in testa alla colonna stessa.

		a)		b)		c)		d)	
		$(x,y) \in \mathbb{R}^2$		$x > 0,\ y > 0$		$0 \leq x \leq y$		$x > 0,\ y \leq x^2$	
	Funzione	liminf	limsup	liminf	limsup	liminf	limsup	liminf	limsup
1)	$\frac{x^2}{x^2+y^2}$	0	1	0	1	0	1/2	0	1
2)	$\frac{y}{x^2+y^2}$	$-\infty$	$+\infty$	0	$+\infty$	$+\infty$	$+\infty$	$-\infty$	1
3)	$\frac{y^2}{x^2+y^2}$	0	1	0	1	$\frac{1}{2}$	1	0	1
4)	$\frac{xy}{x^2+y^2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
5)	$\frac{xy^3}{x^2+y^2}$	0	0	0	0	0	0	0	0
6)	$\frac{x^2+2y^2}{x^2+y^2}$	1	2	1	2	3/2	2	1	2
7)	$\frac{y^2}{\lvert x\rvert+\lvert y\rvert}$	0	0	0	0	0	0	0	0
8)	$\frac{x}{x^4+y^4}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	0	$+\infty$
9)	$\frac{x^2y}{x^4+y^4}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	$-\infty$	1
10)	$\frac{x^3y}{x^4+y^4}$	$-\frac{\sqrt[4]{27}}{4}$	$\frac{\sqrt[4]{27}}{4}$	0	$\frac{\sqrt[4]{27}}{4}$	0	$\frac{1}{2}$	$-\frac{\sqrt[4]{27}}{4}$	0
11)	$\frac{x}{x+y}$	$-\infty$	$+\infty$	0	1	0	$\frac{1}{2}$	$-\infty$	$+\infty$
12)	$\frac{y^2}{x+y}$	$-\infty$	$+\infty$	0	0	0	0	$-\infty$	$+\infty$

1.a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ $(=\cos^2\theta)$ $\quad (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

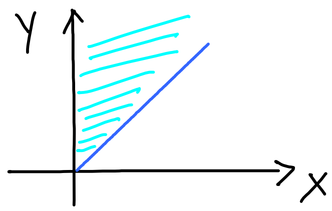
$$\begin{cases} f(x,y)\geq 0 & f(0,\delta)=0 \rightsquigarrow \text{LIMINF}=0 \\ f(x,y)\leq 1 & f(\delta,0)=1 \rightsquigarrow \text{LIMSUP}=1 \end{cases}$$

1.b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ $(=\cos^2\theta)$ $\quad x>0\,,\ y>0$

$$\overset{m,\delta>0}{f(\delta,m\delta)} = \frac{\delta^2}{\delta^2+m^2\delta^2} = \frac{1}{1+m^2}$$

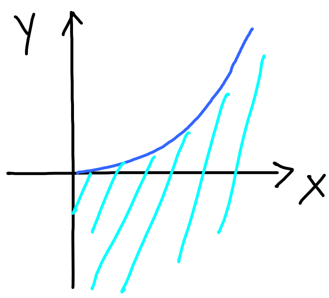
$$\begin{cases} f(x,y)\geq 0 & m \text{ GRANDE} \rightsquigarrow \text{LIMINF}=0 \\ f(x,y)\leq 1 & m \text{ PICCOLO} \rightsquigarrow \text{LIMSUP}=1 \end{cases}$$

1.c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ $(=\cos^2\theta)$ $\quad 0\leq x\leq y$



$$\begin{cases} f(x,y)\geq 0 \quad f(0,\delta)=0 \rightsquigarrow \text{LIMINF}=0 \\ f(x,y)=\dfrac{x^2}{x^2+y^2}\leq\dfrac{x^2}{x^2+x^2}=\dfrac{1}{2} \quad f(\delta,\delta)=\dfrac{1}{2} \rightsquigarrow \text{LIMSUP}=\dfrac{1}{2} \end{cases}$$

1.d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ $(=\cos^2\theta)$ $\quad x>0 \quad y\leq x^2$



$$\overset{m,\delta>0}{f(\delta,-m\delta)} = \frac{\delta^2}{\delta^2+m^2\delta^2} = \frac{1}{1+m^2}$$

$$\begin{cases} f(x,y)\geq 0 & m \text{ GRANDE} \rightsquigarrow \text{LIMINF}=0 \\ f(x,y)\leq 1 & f(\delta,\delta^2)=\dfrac{\delta^2}{\delta^2+\delta^4}=\dfrac{1}{1+\delta^2}\to 1 \rightsquigarrow \text{LIMSUP}=1 \end{cases}$$

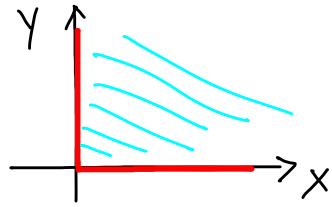
2.a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \left(= \frac{\sin\theta}{\rho}\right) \qquad (x,y)\in\mathbb{R}^2 \setminus \{(0,0)\}$

$$\overset{m>0}{f(\delta, m\delta)} = \frac{m\delta}{\delta^2+m^2\delta^2} = \frac{m}{1+m^2}\,\frac{1}{\delta}$$

$$\lim_{\delta\to0^+} f(\delta, m\delta) = +\infty \rightsquigarrow \text{LIMSUP} = +\infty$$

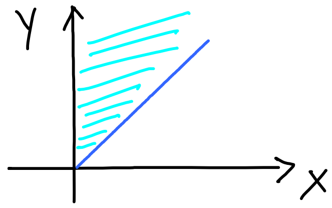
$$\lim_{\delta\to0^-} f(\delta, m\delta) = -\infty \rightsquigarrow \text{LIMINF} = -\infty$$

2.b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \left(= \frac{\sin\theta}{\rho}\right) \qquad x>0,\ y>0$



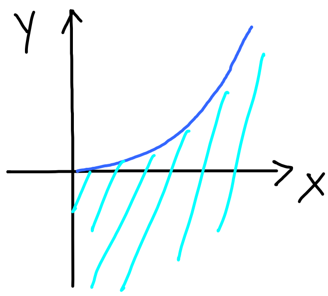
$$\begin{cases} f(\delta,\delta) = \dfrac{\delta}{\delta^2+\delta^2} = \dfrac{1}{2\delta} \to +\infty \quad \delta\to0^+ \rightsquigarrow \text{LIMSUP} = +\infty \\ f(x,y)\geq 0 \quad \overset{a>0}{f(\delta, a\delta^2)} = \dfrac{a\delta^2}{\delta^2+a^2\delta^4} = \dfrac{a}{1+a^2\delta^2} \to a \rightsquigarrow \text{LIMINF} = 0 \end{cases}$$

2.c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \left(= \frac{\sin\theta}{\rho}\right) \qquad 0\leq x\leq y$



$$\frac{y}{x^2+y^2} \geq \frac{y}{y^2+y^2} = \frac{1}{y} \to +\infty$$

2.d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \left(= \frac{\sin\theta}{\rho}\right) \qquad x>0 \quad y\leq x^2$



$$f(\delta, -m\delta) = \frac{-m\delta}{\delta^2+m^2\delta^2} = \frac{1}{\delta}\,\frac{-m}{1+m^2} \to -\infty \quad \delta\to0^+ \rightsquigarrow \text{LIMINF} = -\infty$$

$$f(x,y) = \frac{y}{x^2+y^2} \leq \frac{x^2}{x^2+y^2} = \cos^2\theta \leq 1$$

$$f(\delta,\delta^2) = \frac{\delta^2}{\delta^2+\delta^4} = \frac{1}{1+\delta^2} \to 1 \rightsquigarrow \text{LIMSUP} = 1$$

3.a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} \left(= \sin^2\theta\right) \qquad (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

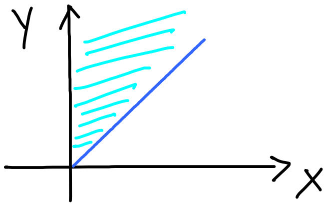
$$\begin{cases} f(x,y)\geq 0 & f(0,\delta)=1 \rightsquigarrow \text{LIMSUP}=1 \\ f(x,y)\leq 1 & f(\delta,0)=0 \rightsquigarrow \text{LIMINF}=0 \end{cases}$$

3.b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} \left(= \sin^2\theta\right) \qquad x>0,\ y>0$

$$f(m\delta,\delta) \overset{m,\delta>0}{=} \frac{\delta^2}{\delta^2+m^2\delta^2} = \frac{1}{1+m^2}$$

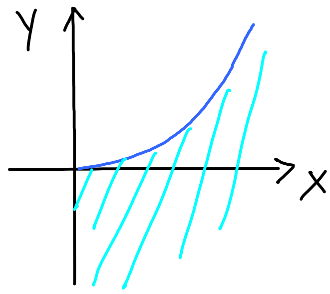
$$\begin{cases} f(x,y)\geq 0 & m \text{ GRANDE} \rightsquigarrow \text{LIMINF}=0 \\ f(x,y)\leq 1 & m \text{ PICCOLO} \rightsquigarrow \text{LIMSUP}=1 \end{cases}$$

3.c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} \left(= \sin^2\theta\right) \qquad 0\leq x\leq y$



$$\begin{cases} f(x,y)\geq \frac{1}{2} & f(\delta,\delta)=\frac{1}{2} \rightsquigarrow \text{LIMINF}=\frac{1}{2} \\ f(x,y)=\frac{y^2}{x^2+y^2}=\sin^2\theta\leq 1 & f(0,\delta)=1 \rightsquigarrow \text{LIMSUP}=1 \end{cases}$$

3.d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} \left(= \sin^2\theta\right) \qquad x>0 \quad y\leq x^2$



$$f(\delta,-m\delta) \overset{m,\delta>0}{=} \frac{m^2\delta^2}{\delta^2+m^2\delta^2} = \frac{m^2}{1+m^2}$$

$$\begin{cases} f(x,y)\leq 1 & m \text{ GRANDE} \rightsquigarrow \text{LIMSUP}=1 \\ f(x,y)\geq 0 & f(\delta,\delta^2)=\frac{\delta^4}{\delta^2+\delta^4}=\frac{\delta^2}{1+\delta^2}\to 0 \rightsquigarrow \text{LIMINF}=0 \end{cases}$$

5.a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \left(=\frac{1}{2}\sin 2\theta\right) \quad (x,y)\in\mathbb{R}^2 \setminus \{(0,0)\}$

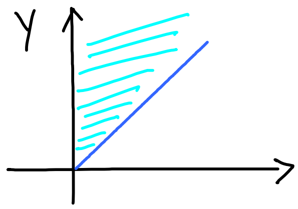
$$-\frac{1}{2} \le \frac{xy}{x^2+y^2} \le \frac{1}{2}$$

$$\begin{cases} f(\delta,\delta) = \frac{1}{2} \rightsquigarrow \text{LIMSUP} = \frac{1}{2} \\ f(\delta,-\delta) = -\frac{1}{2} \rightsquigarrow \text{LIMINF} = -\frac{1}{2} \end{cases}$$

5.b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \left(=\frac{1}{2}\sin 2\theta\right) \quad x>0,\ y>0 \qquad 0 \le \frac{xy}{x^2+y^2} \le \frac{1}{2}$

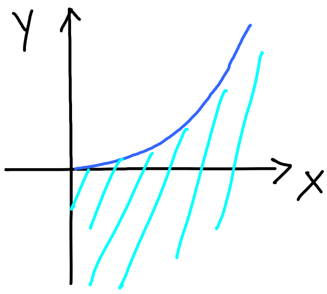
$$f(\delta, \overset{m>0}{m}\delta) = \frac{m\delta^2}{\delta^2+m^2\delta^2} = \frac{m}{1+m^2} \quad \begin{cases} m \text{ PICCOLO} \rightsquigarrow \text{LIMINF} = 0 \\ m = 1 \rightsquigarrow \text{LIMSUP} = \frac{1}{2} \end{cases}$$

5.c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \left(=\frac{1}{2}\sin 2\theta\right) \quad 0 \le x \le y$



$$\begin{cases} f(x,y) \ge 0 \quad f(0,\delta) = 0 \rightsquigarrow \text{LIMINF} = 0 \\ f(x,y) \le \frac{1}{2} \quad f(\delta,\delta) = \frac{1}{2} \rightsquigarrow \text{LIMSUP} = \frac{1}{2} \end{cases}$$

5.d) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \left(=\frac{1}{2}\sin 2\theta\right) \quad x>0 \quad y \le x^2$



$\underline{0 \le y \le x^2}$ $\qquad 0 \le \dfrac{xy}{x^2+y^2} \le \dfrac{x\,x^2}{x^2+0} = x \to 0$

$\underline{x>0,\ y<0}$ $\qquad -\frac{1}{2} \le f(x,y) \le 0 \qquad f(\delta,-\delta) = \dfrac{-\delta^2}{2\delta^2} = -\dfrac{1}{2}$

$\rightsquigarrow$ LIMSUP $= 0$, LIMINF $= -\frac{1}{2}$

5.a) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$ $\left(= \rho^2 \cos\theta \sin^3\theta\right)$ $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

$0 \le \frac{|xy^3|}{x^2+y^2} = \rho^2 |\cos\theta\sin^3\theta| \le \rho^2 \to 0 \rightsquigarrow$ LIMINF = LIMSUP = 0

5.b) 5.c) 5.d) $\rightsquigarrow$ LIMINF = LIMSUP = 0

6.a) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ $\left(= 1+\sin^2\theta\right)$ $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

$1 \le f(x,y) \le 2$

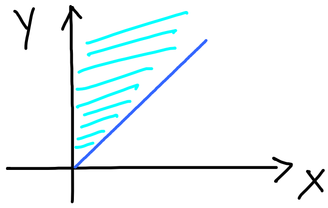
$f(0,\delta) = 2 \rightsquigarrow$ LIMSUP = 2 $\qquad f(\delta,0) = 1 \rightsquigarrow$ LIMINF = 1

6.b) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ $\left(= 1+\sin^2\theta\right)$ $x>0,\ y>0$

$1 \le f(x,y) \le 2$

$f(\delta, \overset{m>0}{m\delta}) = \frac{1+2m^2}{1+m^2} \rightsquigarrow \begin{cases} m\to 0^+ & \rightsquigarrow \text{LIMINF} = 1 \\ m\to+\infty & \rightsquigarrow \text{LIMSUP} = 2 \end{cases}$

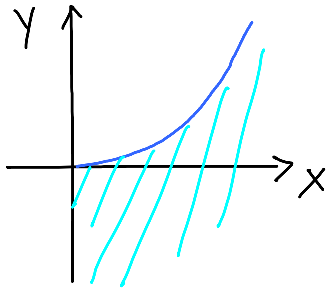
6.c) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ $\left(= 1+\sin^2\theta\right)$ $0\le x\le y$



$\frac{3}{2} \le f(x,y) \le 2$

$f(\delta, \overset{m\ge 1}{m\delta}) = \frac{1+2m^2}{1+m^2} \rightsquigarrow \begin{cases} m=1 & \rightsquigarrow \text{LIMINF} = 3/2 \\ m\to+\infty & \rightsquigarrow \text{LIMSUP} = 2 \end{cases}$

6.d) $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ $\left(= 1+\sin^2\theta\right)$ $x>0\ \ y\le x^2$



$1 \le f(x,y) \le 2$

$f(\delta,0) = 1 \rightsquigarrow$ LIMINF = 1

$f(\delta, \overset{m>0}{-m\delta}) = \frac{1+2m^2}{1+m^2} \rightsquigarrow m\to+\infty \rightsquigarrow$ LIMSUP = 2

7.a) $\lim_{(x,y)\to(0,0)} \frac{y^2}{|x|+|y|} \quad \left(= \frac{\rho \sin^2\theta}{|\cos\theta|+|\sin\theta|}\right) \quad (x,y)\in\mathbb{R}^2 \setminus \{(0,0)\}$

$$0 \le \frac{y^2}{|x|+|y|} \le \frac{y^2}{|y|} = |y| \to 0 \quad \rightsquigarrow \quad \text{LIMINF} = \text{LIMSUP} = 0$$

7.b) 7.c) 7.d) $\rightsquigarrow$ LIMINF = LIMSUP = 0

8.a) $\lim_{(x,y)\to(0,0)} \frac{x}{x^5+y^5} \quad \left(= \frac{\cos\theta}{\rho^3(\cos^5\theta+\sin^5\theta)}\right) \quad (x,y)\in\mathbb{R}^2 \setminus \{(0,0)\}$

$$f(\delta,\delta) = \frac{1}{2\delta^3} \to \begin{cases} +\infty & \delta\to 0^+ \rightsquigarrow \text{LIMSUP} = +\infty \\ -\infty & \delta\to 0^- \rightsquigarrow \text{LIMINF} = -\infty \end{cases}$$

8.b) $\lim_{(x,y)\to(0,0)} \frac{x}{x^5+y^5} \quad \left(= \frac{\cos\theta}{\rho^3(\cos^5\theta+\sin^5\theta)}\right) \quad x>0,\ y>0$

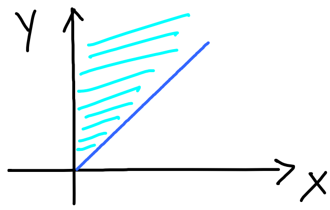
$a>0$

$$f(a\delta^5, \delta) = \frac{a\delta^5}{a^5\delta^{16}+\delta^5} = \frac{a}{a^5\delta^{12}+1} \to a \quad \rightsquigarrow \text{LIMINF} = 0$$

$m>0$

$$f(\delta, m\delta) = \frac{1}{\delta^3}\,\frac{1}{1+m^5} \to +\infty \quad \rightsquigarrow \text{LIMSUP} = +\infty$$

8.c) $\lim_{(x,y)\to(0,0)} \frac{x}{x^5+y^5} \quad \left(= \frac{\cos\theta}{\rho^3(\cos^5\theta+\sin^5\theta)}\right) \quad 0 \le x \le y$



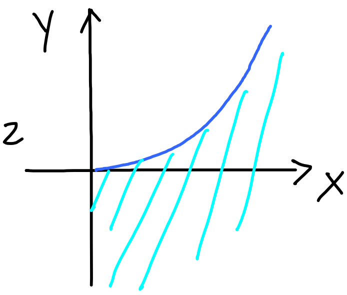
$a>0$

$$f(a\delta^5, \delta) = \frac{a\delta^5}{a^5\delta^{16}+\delta^5} = \frac{a}{a^5\delta^{12}+1} \to a \quad \rightsquigarrow \text{LIMINF} = 0$$

$m>1$

$$f(\delta, m\delta) = \frac{1}{\delta^3}\,\frac{1}{1+m^5} \to +\infty \quad \rightsquigarrow \text{LIMSUP} = +\infty$$

8.d) $\lim_{(x,y)\to(0,0)} \frac{x}{x^5+y^5} \quad \left(= \frac{\cos\theta}{\rho^3(\cos^5\theta+\sin^5\theta)}\right) \quad x>0 \quad y \le x^2$



$a>0$

$$f(a\delta^5, -\delta) = \frac{a\delta^5}{a^5\delta^{16}+\delta^5} = \frac{a}{a^5\delta^{12}+1} \to a \quad \rightsquigarrow \text{LIMINF} = 0$$

$$f(\delta, 0) = \frac{1}{\delta^3} \to +\infty \quad \rightsquigarrow \text{LIMSUP} = +\infty$$

3.a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^4} \left(= \dfrac{\cos^2\theta \sin\theta}{\rho(\cos^4\theta+\sin^4\theta)}\right)$ $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

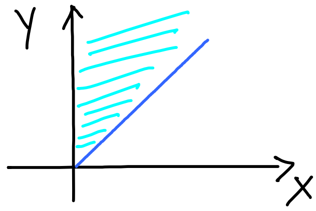
$$f(\delta,\delta) = \frac{1}{2\delta} \to \begin{cases} +\infty & \delta\to0^+ \rightsquigarrow \text{LIMSUP}=+\infty \\ -\infty & \delta\to0^- \rightsquigarrow \text{LIMINF}=-\infty \end{cases}$$

3.b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^4} \left(= \dfrac{\cos^2\theta \sin\theta}{\rho(\cos^4\theta+\sin^4\theta)}\right)$ $x>0,\ y>0$

$$f(x,y)\geq 0,\ f(\delta, \overset{a>0}{a}\delta^2) = \frac{a}{1+a^4\delta^4} \to a \rightsquigarrow \text{LIMINF}=0$$

$$f(\delta,\delta) = \frac{1}{2\delta} \to +\infty \rightsquigarrow \text{LIMSUP}=+\infty$$

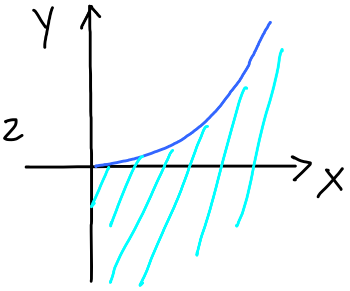
3.c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^4} \left(= \dfrac{\cos^2\theta \sin\theta}{\rho(\cos^4\theta+\sin^4\theta)}\right)$ $0\leq x\leq y$



$$f(x,y)\geq 0,\ f(\delta^2,\delta) = \frac{\delta^5}{\delta^8+\delta^4} = \frac{\delta}{\delta^4+1} \to 0 \rightsquigarrow \text{LIMINF}=0$$

$$f(\delta,\delta) = \frac{1}{2\delta} \to +\infty \rightsquigarrow \text{LIMSUP}=+\infty$$

3.d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^4} \left(= \dfrac{\cos^2\theta \sin\theta}{\rho(\cos^4\theta+\sin^4\theta)}\right)$ $x>0\ \ y\leq x^2$



$\underline{0\leq y\leq x^2}$ $\qquad 0 \leq \dfrac{x^2y}{x^4+y^4} \leq \dfrac{x^4}{x^4+y^4} \leq 1$

$$f(\delta,\delta^2) = \frac{\delta^4}{\delta^4+\delta^8} = \frac{1}{1+\delta^4} \to 1 \rightsquigarrow \text{LIMSUP}=1$$

$\underline{x>0,\ y<0}$ $\qquad f(\delta,-\delta) = \dfrac{-\delta^3}{\delta^4+\delta^4} = \dfrac{-1}{2\delta} \to -\infty$

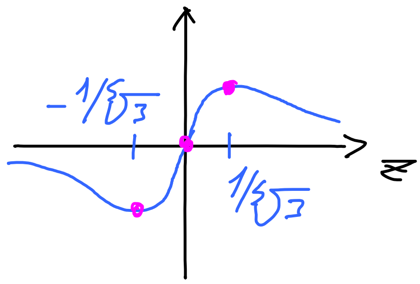
$$\rightsquigarrow \text{LIMINF}=-\infty$$

10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4+y^4} \quad \left(= \frac{\cos^3\theta \,\sin\theta}{(\cos^4\theta+\sin^4\theta)} = \frac{\operatorname{tg}\theta}{1+\operatorname{tg}^4\theta}, \ \cos\theta \neq 0 \right)$

$z = \operatorname{tg}\theta \in (-\infty, +\infty)$

$g(z) = \dfrac{z}{1+z^4} \qquad g'(z) = \dfrac{1+z^4-4z^4}{(1+z^4)^2} = \dfrac{1-3z^4}{(1+z^4)^2}$

$g'(z)$ $\qquad$ $- - - - \circ + + + + \circ - - - - - - -$ (zeros at $-1/\sqrt[4]{3}$ and $1/\sqrt[4]{3}$)



$g(1/\sqrt[4]{3}) = \dfrac{1/\sqrt[4]{3}}{1+\frac{1}{3}} = \dfrac{3}{4\sqrt[4]{3}} = \dfrac{\sqrt[4]{27}}{4}$

$\rightsquigarrow \quad -\dfrac{\sqrt[4]{27}}{4} \leq f(x,y) \leq \dfrac{\sqrt[4]{27}}{4}$

a) $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ $\qquad$ LIMINF $= -\dfrac{\sqrt[4]{27}}{4}$ $\qquad$ LIMSUP $= \dfrac{\sqrt[4]{27}}{4}$

b) $x>0, \ y>0$ $\qquad$ LIMINF $= 0$ $\qquad$ LIMSUP $= \dfrac{\sqrt[4]{27}}{4}$

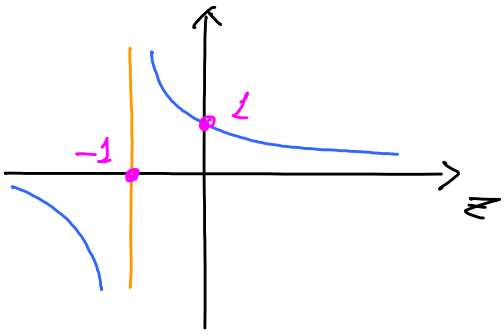
c) $0 \leq x \leq y$ $\qquad$ LIMINF $= 0$ $\qquad$ LIMSUP $= \dfrac{1}{2}$

d) $x>0 \ \ y \leq x^2$ $\qquad$ LIMINF $= -\dfrac{\sqrt[4]{27}}{4}$ $\qquad$ LIMSUP $= 0$

11) $\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$ $\left(= \frac{\cos\theta}{\cos\theta + \sin\theta} = \frac{1}{1+\operatorname{tg}\theta}, \quad \cos\theta \neq 0 \right)$

$z = \operatorname{tg}\theta \in (-\infty, +\infty)$

$g(z) = \frac{1}{1+z} \qquad g'(z) = \frac{-1}{(1+z)^2}$



a) $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ LIMINF $= -\infty$ LIMSUP $= +\infty$

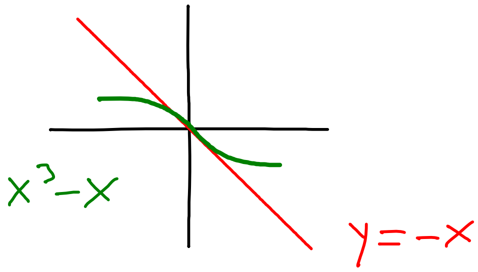
b) $x > 0, \ y > 0$ LIMINF $= 0$ LIMSUP $= 1$

c) $0 \le x \le y$ LIMINF $= 0$ LIMSUP $= 1/2$

d) $x > 0 \ \ y \le x^2$ LIMINF $= -\infty$ LIMSUP $= +\infty$

12) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2}{x+y} \quad \left(= \dfrac{\rho \sin^2\theta}{\cos\theta+\sin\theta}\right)$

a) $(x,y)\in\mathbb{R}^2 \quad y \neq -x$



$$f(\delta,\delta^3-\delta)=\frac{(\delta^3-\delta)^2}{\delta^3}=\frac{(\delta^2-1)^2}{\delta}\to +\infty \qquad \delta\to 0^+$$

$$\to -\infty \qquad \delta\to 0^-$$

$\rightsquigarrow$ LIMINF $= -\infty$ LIMSUP $= +\infty$

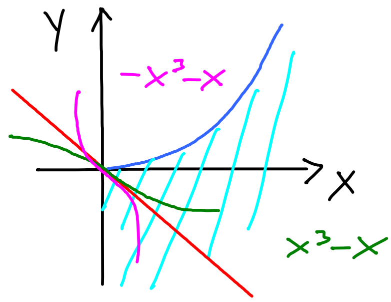
b) $x>0,\ y>0 \qquad 0\le \dfrac{y^2}{|x+y|}=\dfrac{\rho\sin^2\theta}{|\cos\theta+\sin\theta|}\le \dfrac{\rho\cdot 1}{\underbrace{m}_{>0}}\to 0$

$\rightsquigarrow$ LIMINF $= 0$ LIMSUP $= 0$

c) $0\le x\le y \qquad 0\le \dfrac{y^2}{|x+y|}=\dfrac{\rho\sin^2\theta}{|\cos\theta+\sin\theta|}\le \dfrac{\rho\cdot 1}{\underbrace{m}_{>0}}\to 0$

$\rightsquigarrow$ LIMINF $= 0$ LIMSUP $= 0$

d) $x>0 \quad y\le x^2$



$$f(\delta,\delta^3-\delta)=\frac{(\delta^3-\delta)^2}{\delta^3}=\frac{(\delta^2-1)^2}{\delta}\to +\infty \qquad \delta\to 0^+$$

$$f(\delta,-\delta^3-\delta)=\frac{(-\delta^3-\delta)^2}{-\delta^3}=\frac{(-\delta^2-1)^2}{-\delta}\to -\infty \qquad \delta\to 0^+$$

$\rightsquigarrow$ LIMINF $= -\infty$ LIMSUP $= +\infty$