Università di Pisa - Corso di Laurea in Matematica

Exam paper of “Elementi di Calcolo delle Variazioni”

Pisa, 23 September 2017

1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\dot{u}^2 + u^2 - x^2 \dot{u}\right)\, dx.$$

Discuss the minimum problem for $F(u)$ subject to each of the following boundary conditions:

(a) $u(1) = 2u(-1)$,

(b) $u(0) = 0$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{u + u^3}{1 + \dot{u}^4}, \qquad u(0) = 1, \quad u(1) = \lambda.$$

(a) Discuss existence, uniqueness, and regularity of the solution.

(b) Determine the values of λ for which the solution is convex.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^{\ell} \left[(1 + u^2)\dot{u}^2 - 10 \sin^2(u) + \cos x \cdot u^6\right]\, dx : u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the infimum is actually a minimum.

(b) Determine for which values of ℓ the minimum exists and is negative.

4. Let us consider, for every $\varepsilon > 0$, the problem

$$m_\varepsilon = \inf \left\{ \int_0^{2} \left(\varepsilon \dot{u}^4 - \dot{u}^2 + \varepsilon^2 u^4\right)\, dx : u(0) = u(2) = 2 \right\}.$$

(a) Determine for which values $\varepsilon > 0$ the infimum is a real number.

(b) Determine for which values $\varepsilon > 0$ the infimum is actually a minimum.

(c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.