Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 27 July 2017

1. Discuss the minimum problem

$$\min\left\{\int_0^1 \left(\dot{u}-x^2\right)^2\,dx : u(0)=u(1)\right\}.$$

2. Discuss existence, uniqueness, and regularity of the solution to the boundary value problem

$$\ddot{u}=\frac{u^3-x^3}{1+\dot{u}^2}, \qquad u(0)=1, \quad \dot{u}(1)=3.$$

3. Let us consider, for every $\ell>0$, the problem

$$\inf\left\{\int_0^\ell (-\cos(\dot{u})+\cos(u))\,dx : u(0)=u(\ell)=0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a strong local minimum.

   (c) Determine the infimum as a function of ℓ.

4. Let us consider, for every $\varepsilon>0$, the problem

$$m_\varepsilon=\min\left\{\int_0^1 \left(\sinh(\dot{u}^2)+u^6\right)\,dx : u(0)=u(1)=0,\ \int_0^1 u^4\,dx=\varepsilon\right\}.$$

   (a) Prove that the minimum exists for every $\varepsilon>0$.

   (b) Determine all real numbers α for which

$$\lim_{\varepsilon\to 0^+}\frac{m_\varepsilon}{\varepsilon^\alpha}=0.$$

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.