Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 26 June 2017

1. Let us consider the functional

$$F(u) = \int_0^\pi \left[\ddot{u}^2 + \dot{u}^2 + \sin x \cdot u\right] dx.$$

Discuss the minimum problem for $F(u)$ subject to each of the following boundary conditions:

(a) $u(0) = \pi$,

(b) $u'(0) = 2017$.

2. Let us consider the boundary value problem

$$\ddot{u} = \arctan x \cdot \arctan u, \qquad u(0) = \pi, \quad u(\pi) = 0.$$

(a) Discuss existence, uniqueness, and regularity of the solution.

(b) Discuss the monotonicity of the solution.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left[\arctan(\dot{u}^2) + \sin(u^2)\right] dx : u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

4. For every positive integer n, let us set

$$M_n = \inf \left\{ \int_0^1 \left(2017\dot{u}^2 + nu^6 - n^2u^2\right) dx : u(0) = u(1) = 0 \right\}.$$

(a) Determine for which values of n the infimum is actually a minimum.

(b) Determine for which values of n the infimum is negative.

(c) Determine the leading term of M_n as $n \to +\infty$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.