Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 5 June 2017

1. Discuss the minimum problem

$$\min\left\{\int_0^1 \left[(\dot u + x)^2 + (u+x)^2\right]\, dx : u'(1) = \alpha\right\}$$

depending on the real parameter α.

2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot u = \frac{u^3}{u^2+1}\cdot|\cos x|, \qquad u'(0) = 1, \quad u(1) = 0.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\min\left\{\int_0^\ell \left[\sinh(\dot u^2) - \sin(u^2) + u^4\right]\, dx : u(0) = u(\ell) = 0\right\}.$$

   (a) Determine for which values of ℓ the minimum exists.

   (b) Determine for which values of ℓ the minimum (exists and) is negative.

   (c) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

4. Let us set

$$m_\varepsilon = \inf\left\{\int_0^1 \left(\varepsilon\dot u^2 + \arctan\dot u + \arctan u\right)\, dx : u(0) = 3,\ u(1) = 2017\right\},$$

where ε is a positive real parameter.

   (a) Determine if there exist values of ε for which the infimum is actually a minimum.

   (b) Determine if there exist values of ε for which the infimum is not a minimum.

   (c) Compute the limit of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.