Università di Pisa - Corso di Laurea in Matematica

Scritto d'esame di Elementi di Calcolo delle Variazioni

Pisa, 24 February 2017

1. Discuss the minimum problem

$$\min\left\{\int_0^\ell \left(\dot{u}^2 + u\sin x\right)\,dx : u(0) = u(\ell)\right\}$$

depending on the parameter $\ell > 0$.

2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{u-3}{3+\cos\dot{u}}, \qquad u(0) = 2017, \quad u'(2017) = 0.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left(\sin(\dot{u}^2) - \sinh(u^2)\right)\,dx : u(0) = u(\ell) = 0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Compute the infimum as a function of ℓ.

4. Let us set

$$m_\varepsilon = \inf\left\{\int_0^1 \left(\varepsilon\dot{u}^4 - \dot{u}^2 + u^2\right)\,dx : u(0) = u(1) = 2017\right\},$$

where ε is a positive real parameter.

   (a) Determine for which values of ε it turns out that $m_\varepsilon \in \mathbb{R}$.

   (b) Determine for which values of ε the infimum is actually a minimum.

   (c) Determine the leading term of m_ε as $\varepsilon \to 0^+$.

Every step has to be *reasonably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.