Siano a, b, c tre numeri reali strettamente positivi.

Studiare, al variare di $a, b, c$ la convergenza della serie,

$$\sum_{n=1}^{+\infty}\left(\sqrt[n]{a}-\frac{\sqrt[n]{b}+\sqrt[n]{c}}{2}\right)$$

$$\sqrt[n]{a} = e^{\log \sqrt[n]{a}} = e^{\frac{\log a}{n}} = 1 + \frac{\log a}{n} + o\left(\frac{1}{n}\right)$$

$$\sqrt[n]{a} - \frac{1}{2}\sqrt[n]{b} - \frac{1}{2}\sqrt[n]{c} = \cancel{1} + \frac{\log a}{n} - \cancel{\frac{1}{2}} - \frac{1}{2}\frac{\log b}{n} - \cancel{\frac{1}{2}} - \frac{1}{2}\frac{\log c}{n} + o\left(\frac{1}{n}\right) =$$

$$= \left(\log a - \frac{1}{2}\log bc\right)\frac{1}{n} + o\left(\frac{1}{n}\right) = \frac{1}{n}\log\frac{a}{\sqrt{bc}} + o\left(\frac{1}{n}\right)$$

$\Rightarrow$ $\forall\, a, b, c > 0$ T.C. $a \neq \sqrt{bc}$ PER CONFRONTO ASINTOTICO CON $\sum \frac{1}{n}$ LA SERIE DATA DIVERGE

$\forall\, a, b, c > 0$ T.C. $a = \sqrt{bc}$ PER CONFRONTO ASINTOTICO CON $\sum \frac{1}{n^2}$ LA SERIE DATA CONVERGE

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CONGETTURA $\quad \sum\left(\sqrt[n]{a} - \frac{\sqrt[n]{b}+\sqrt[n]{c}}{2}\right) = 0 \iff a = b = c$

$\underline{a = b = c} \Rightarrow \sum = 0$

$\underline{a \neq \sqrt{bc}} \Rightarrow \sum = +\infty$

$\underline{a = \sqrt{bc}}$ $\qquad \sqrt[n]{a} = 1 + \frac{\log a}{n} + \frac{1}{2}\frac{\log^2 a}{n^2} + o\left(\frac{1}{n^2}\right)$

$$\sqrt[n]{a} - \frac{1}{2}\sqrt[n]{b} - \frac{1}{2}\sqrt[n]{c} = \cancel{1} + \cancel{\frac{\log a}{n}} + \frac{1}{2}\frac{\log^2 a}{n^2} - \cancel{\frac{1}{2}} - \frac{1}{2}\cancel{\frac{\log b}{n}} - \frac{1}{4}\frac{\log^2 b}{n^2} +$$

$$- \cancel{\frac{1}{2}} - \frac{1}{2}\cancel{\frac{\log c}{n}} - \frac{1}{4}\frac{\log^2 c}{n^2} + o\left(\frac{1}{n^2}\right) =$$

$$= \frac{1}{8n^2}\left(4\log^2\sqrt{bc} - 2\log^2 b - 2\log^2 c\right) + o\left(\frac{1}{n^2}\right)$$

$$4\log^2\sqrt{bc} - 2\log^2 b - 2\log^2 c =$$

$$= \log^2 bc - 2\log^2 b - 2\log^2 c =$$

$$= (\log b + \log c)^2 - 2\log^2 b - 2\log^2 c =$$

$$= \cancel{\log^2 b} + \cancel{\log^2 c} + 2\log b \cdot \log c - \cancel{2}\log^2 b - \cancel{2}\log^2 c =$$

$$= -(\log b - \log c)^2 = -\log^2 \frac{b}{c} = 0 \iff b = c$$

$$\Rightarrow \quad a = \sqrt{bc} = b = c$$

$$\rightsquigarrow \quad \sum \left( \sqrt[n]{a} - \frac{\sqrt[n]{b} + \sqrt[n]{c}}{2} \right) = 0 \iff a = b = c$$

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<u>MODO PIÚ FURBO</u>

DISEG. AM-GM : $\dfrac{x+y}{2} \geq \sqrt{xy} \quad \forall x, y \in [0, +\infty)$

PER $a = \sqrt{bc}$

$$\sqrt[n]{a} = \sqrt[n]{\sqrt{bc}} = \sqrt{\sqrt[n]{bc}} = \sqrt{\sqrt[n]{b}\,\sqrt[n]{c}} \leq \frac{\sqrt[n]{b} + \sqrt[n]{c}}{2}$$

CON UGUAGLIANZA SE E SOLO SE $a = b = c$