Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Prova in Itinere di Analisi Matematica II

Pisa, ?? ?? ????

1. Sia D il dominio del piano xy delimitato dalla curva $(\log t, t + t^2)$ con $1 \leq t \leq 2$, dalla retta $x = 0$ e dalla retta $y = 6$.

   (a) Fare un disegno approssimativo di D.

   (b) Calcolare l'area di D.

2. Sia S la superficie data da $S = \{(x, y, z) \in \mathbb{R}^3 : \; x^2 - y + z^2 = 4, \; -1 \leq y \leq 0\}$, orientata prendendo nel punto $(2, 0, 0)$ la normale che punta verso le x positive. Sia $F(x, y, z) = (x^2, e^{xy}, z)$. Calcolare il flusso del rotore di F attraverso S.

3. Sia $B := \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \geq 1, \; x \geq 0, \; y \geq 0\}$. Si consideri

$$I_\alpha := \int_B \frac{\arctan(xy)}{x^\alpha + y^\alpha} dx\, dy$$

   (a) Provare che per $\alpha = 4$ l'integrale I_4 converge.

   (b) Stabilire per quali valori del parametro $\alpha > 0$ l'integrale I_α converge.

Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato.
Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

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   (a) Fare un disegno approssimativo di D.

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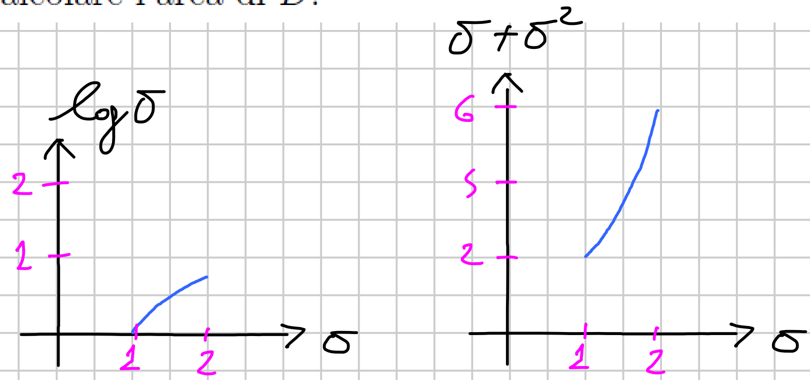
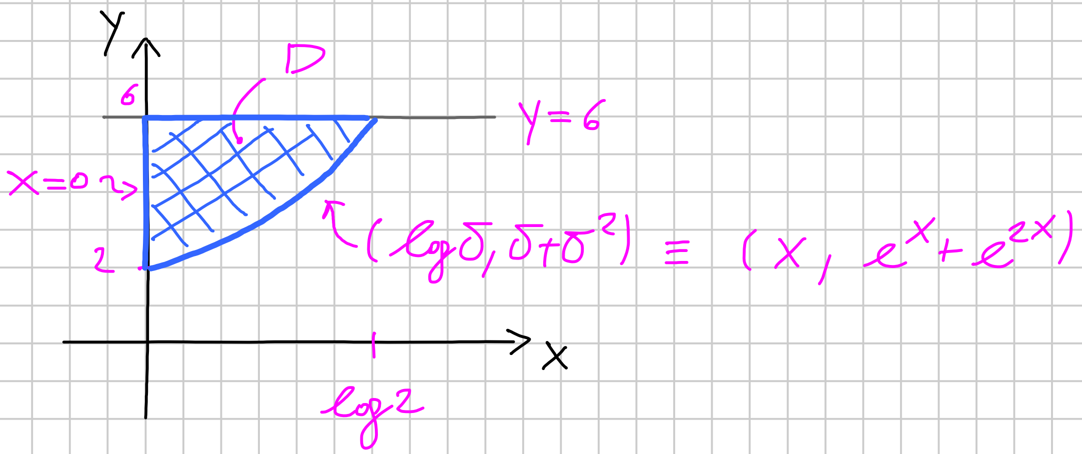
(a)





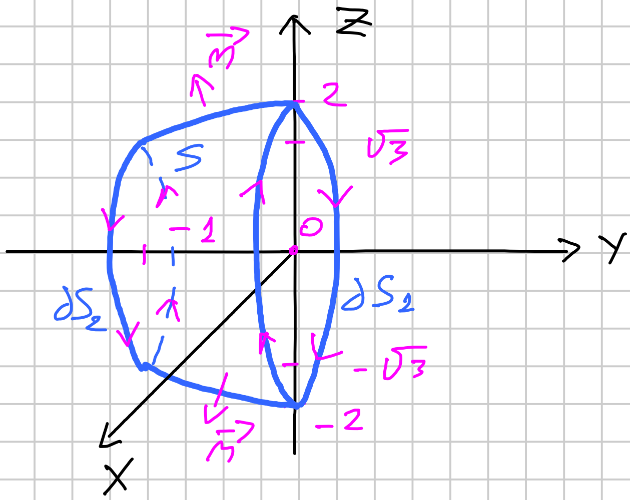
(b)

$$A_D = \int_0^{\log 2} (6 - e^x - e^{2x})\, dx =$$

$$= \left[6x - e^x - \frac{1}{2} e^{2x} \right]_0^{\log 2} = 6 \log 2 - 2 - 2 + 1 + \frac{1}{2} =$$

$$= 6 \log 2 - \frac{5}{2}$$

2. Sia S la superficie data da $S = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 - y + z^2 = 4, \ -1 \le y \le 0\}$, orientata prendendo nel punto $(2,0,0)$ la normale che punta verso le x positive. Sia $F(x,y,z) = (x^2, e^{xy}, z)$. Calcolare il flusso del rotore di F attraverso S.



MODO 1 - TEOREMA DI STOKES

$$\int_S \operatorname{rot} \vec{F} \cdot \vec{n} \, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{\tau} \, ds \qquad \partial S = \partial S_1 \cup \partial S_2$$

$$\partial S_1 = \{ (2\cos\theta, 0, 2\sin\theta), \ \theta \in [0, 2\pi] \}$$

$$\partial S_2 = \{ (\sqrt{3}\cos\theta, -1, -\sqrt{3}\sin\theta), \ \theta \in [0, 2\pi] \}$$

$$\int_{\partial S} \vec{F} \cdot \vec{\tau} \, ds = \int_{\partial S_1} \vec{F} \cdot \vec{\tau} \, ds + \int_{\partial S_2} \vec{F} \cdot \vec{\tau} \, ds$$

$$\int_{\partial S_1} \vec{F} \cdot \vec{\tau} \, ds = \int_0^{2\pi} x^2 dx + z \, dz =$$

$$= \int_0^{2\pi} [4\cos^2\theta(-2\sin\theta) + 2\sin\theta(2\cos\theta)] \, d\theta =$$

$$= \int_0^{2\pi} -8\cos^2\vartheta \sin\vartheta + 2\sin 2\vartheta \, d\vartheta =$$

$$= -8\int_0^{2\pi} \cos^2\vartheta \sin\vartheta \, d\vartheta + 2\overset{=0}{\int_0^{2\pi} \cancel{\sin 2\vartheta} \, d\vartheta} =$$

$$= -8\left[-\frac{1}{3}\cos^3\vartheta\right]_0^{2\pi} = \frac{8}{3}(1-1) = 0$$

$$\int_{\partial S_2} \vec{F}\cdot\vec{T}\, ds = \int_0^{2\pi} x^2 dx + z\, dz =$$

$$= \int_0^{2\pi} \left[3\cos^2\vartheta(-\sqrt{3}\sin\vartheta) - \sqrt{3}\sin\vartheta(\sqrt{3}\cos\vartheta)\right] d\vartheta =$$

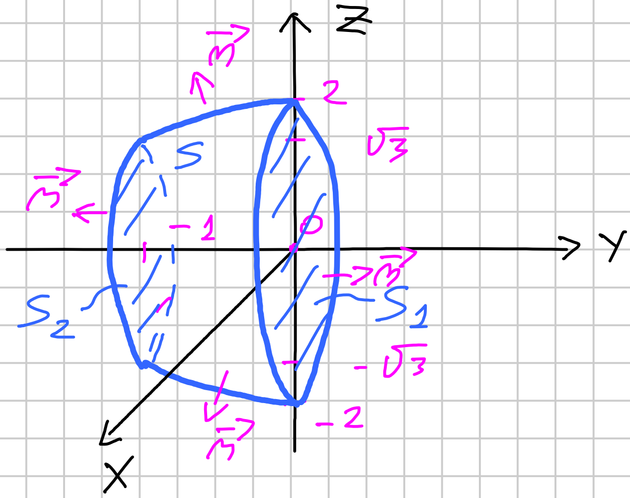
$$= \int_0^{2\pi} -3\sqrt{3}\cos^2\vartheta \sin\vartheta \, d\vartheta - \frac{3}{2}\int_0^{2\pi} \sin 2\vartheta \, d\vartheta = 0$$

$$\leadsto \int_S \text{rot}\,\vec{F}\cdot\vec{n}\, d\sigma = \int_{\partial S} \vec{F}\cdot\vec{T}\, ds = 0 + 0 = 0$$

MODO 2 – TEOREMA DI G.G. + CALCOLO FLUSSO

$$\text{rot}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ x^2 & e^{xy} & z \end{vmatrix} = 0\cdot\hat{i} + 0\cdot\hat{j} + y e^{xy}\hat{k}$$

$$\text{div}(\text{rot}\,\vec{F}) = 0$$



$$\int_{S\cup S_1\cup S_2} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma = \int_{\Omega} \operatorname{div}(\operatorname{rot}\vec{F})\,dx\,dy\,dz = 0$$

$$\int_{S} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma = -\int_{S_1} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma - \int_{S_2} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma$$

$$\int_{S_1} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma = \int_{S_2} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma = 0 \quad (\operatorname{rot}\vec{F}\perp\vec{n})$$

$$\rightsquigarrow \int_{S} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma = 0 + 0 = 0$$

3. Sia $B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1,\ x \geq 0,\ y \geq 0\}$. Si consideri

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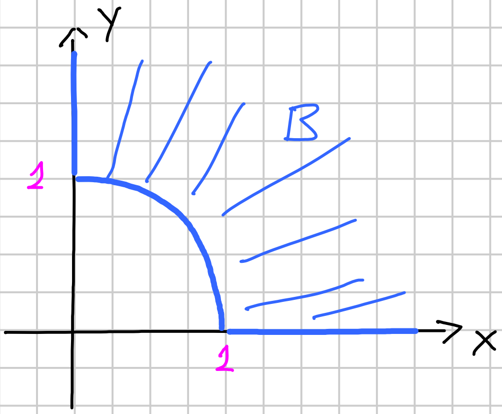

(a) $$I_4 = \int_B \frac{\arctan(xy)}{x^4+y^4}\,dx\,dy =$$ COORD. POLARI $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

$$= \int_0^{\pi/2}\int_1^{+\infty} \frac{\cancel{\rho}\,\overbrace{\arctan(\rho^2\cos\theta\sin\theta)}^{\leq \pi/2}}{\rho^{\cancel{4}\,3}(\cos^4\theta + \sin^4\theta)}\,d\rho\,d\theta \leq$$

$0 < c \leq \cos^4\theta + \sin^4\theta \leq 1$

$$\leq \int_0^{\pi/2}\int_1^{+\infty} \frac{\pi/2}{\rho^3 \cdot c}\,d\rho\,d\theta < +\infty \rightsquigarrow \text{CONVERGE}$$

(b) $$I_\alpha = \int_B \frac{\arctan(xy)}{x^\alpha + y^\alpha}\,dx\,dy =$$

$$= \int_0^{\pi/2}\int_1^{+\infty} \frac{\arctan(\rho^2\cos\theta\sin\theta)}{\rho^{\alpha-1}(\cos^\alpha\theta + \sin^\alpha\theta)}\,d\rho\,d\theta$$

TUTTE Q.TÀ NON NEGATIVE

$\underline{0<\alpha\le 2}$

$$I_\alpha = \int_0^{\pi/2}\int_1^{+\infty} \frac{\arctan(\rho^2\cos\theta\sin\theta)}{\rho^{\alpha-1}(\cos^2\theta+\sin^2\theta)}\, d\rho\, d\theta \ge$$

$\ge m = \arctan(\sqrt{3}/4)$

$$\ge \int_{\pi/6}^{\pi/4}\int_1^{+\infty} \frac{\arctan(\rho^2\cos\theta\sin\theta)}{\rho^{\alpha-1}(\cos^2\theta+\sin^2\theta)}\, d\rho\, d\theta \ge$$

$0<c\le \cos^2\theta+\sin^2\theta \le M=2$

$$\ge \int_{\pi/6}^{\pi/4}\int_1^{+\infty} \frac{m}{\rho^{\alpha-1}\cdot M}\, d\rho\, d\theta = \frac{\pi}{12}\,\frac{m}{M}\int_1^{+\infty}\frac{1}{\rho^{\alpha-1}}\, d\rho = +\infty$$

TUTTE Q.TÀ NON NEGATIVE

$\underline{\alpha>2}$

$$I_\alpha = \int_0^{\pi/2}\int_1^{+\infty} \frac{\arctan(\rho^2\cos\theta\sin\theta)}{\rho^{\alpha-1}(\cos^2\theta+\sin^2\theta)}\, d\rho\, d\theta \le$$

$$\le \int_0^{\pi/2}\int_{1/2}^{+\infty} \frac{\arctan(\rho^2\cos\theta\sin\theta)}{\rho^{\alpha-1}(\cos^2\theta+\sin^2\theta)}\, d\rho\, d\theta \le$$

$$\le \int_0^{\pi/2}\int_{1/2}^{+\infty} \frac{\pi/2}{\rho^{\alpha-1}\; c}\, d\rho\, d\theta = \frac{1}{c}\left(\frac{\pi}{2}\right)^2\int_{1/2}^{+\infty}\frac{1}{\rho^{\alpha-1}}\, d\rho < +\infty$$

$$\leadsto I_\alpha \begin{cases} \text{CONVERGE PER } \alpha>2 \\ \text{DIVERGE PER } 0<\alpha\le 2 \end{cases}$$