BY GIMUSI 25/01/2015

x^2

Lim x->+inf (1~~/x~~^2)[1/e (1+ 4/x)^(x/4) - (2/pi) arctan (x/pi)]

$$\lim_{x \to +\infty} x^2 \left[\frac{1}{e} \left(1 + \frac{4}{x}\right)^{x/4} - \frac{2}{\pi} \text{ARCTAN}(x/\pi) \right] = \frac{22}{3}$$

$$\left(1 + \frac{4}{x}\right)^{x/4} = e^{\frac{x}{4} \log\left(1 + \frac{4}{x}\right)}$$

$$\frac{x}{4} \log\left(1 + \frac{4}{x}\right) = \frac{x}{4}\left(\frac{4}{x} - \frac{1}{2}\left(\frac{4}{x}\right)^2 + \frac{1}{3}\left(\frac{4}{x}\right)^3 + o\left(\frac{1}{x^3}\right)\right) =$$

$$= 1 - \frac{1}{2}\left(\frac{4}{x}\right) + \frac{1}{3}\left(\frac{4}{x}\right)^2 + o\left(\frac{1}{x^2}\right)$$

$$\left(1 + \frac{4}{x}\right)^{x/4} = e^{\left[1 - \frac{1}{2}\left(\frac{4}{x}\right) + \frac{1}{3}\left(\frac{4}{x}\right)^2 + o\left(\frac{1}{x^2}\right)\right]} =$$

$$= e\left\{1 - \frac{1}{2}\left(\frac{4}{x}\right) + \frac{1}{3}\left(\frac{4}{x}\right)^2 + \frac{1}{2}\left[-\frac{1}{2}\left(\frac{4}{x}\right) + \frac{1}{3}\left(\frac{4}{x}\right)^2\right]^2 + o\left(\frac{1}{x^2}\right)\right\} =$$

$$= e\left[1 - \frac{2}{x} + \frac{16}{3}\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right)^2 + o\left(\frac{1}{x^2}\right)\right]$$

$$\text{ARCTAN}(x/\pi) = \frac{\pi}{2} - \text{ARCTAN}\left(\frac{\pi}{x}\right)$$

$$(2/\pi)\, \text{ARCTAN}(x/\pi) = 1 - \frac{2}{\pi} \text{ARCTAN}\left(\frac{\pi}{x}\right)$$

$$\text{ARCTAN}\left(\frac{\pi}{x}\right) = \frac{\pi}{x} - \frac{1}{3}\left(\frac{\pi}{x}\right)^3 + o\left(\frac{1}{x^3}\right)$$

$$(2/\pi)\, \text{ARCTAN}(x/\pi) = 1 - \frac{2}{x} + \frac{2\pi^2}{3}\left(\frac{1}{x}\right)^3 + o\left(\frac{1}{x^3}\right) =$$

$$= 1 - \frac{2}{x} + o\left(\frac{1}{x^2}\right)$$

$$x^2\left[\frac{1}{e}\left(1+\frac{4}{x}\right)^{x/4}-\frac{2}{\pi}\arctan(x/\pi)\right]=$$

$$=x^2\left[\cancel{1}-\cancel{\frac{2}{x}}+\frac{22}{3}\left(\frac{1}{x}\right)^2+o\left(\frac{1}{x^2}\right)-\cancel{1}+\cancel{\frac{2}{x}}+o\left(\frac{1}{x^2}\right)\right]=$$

$$=x^2\left[\frac{22}{3}\left(\frac{1}{x}\right)^2+o\left(\frac{1}{x^2}\right)\right]=\frac{22}{3}+\overbrace{x^2\,o\left(\frac{1}{x^2}\right)}^{\to 0}\to\frac{22}{3}$$