

## Funzioni trigonometriche inverse 1

Argomenti: principio di induzione

Difficoltà: \*\*\*

Prerequisiti: principio di induzione

1. Completare la seguente tabella

$\arcsin(-1)$	$-\pi/2$	$\arccos(-1)$	$\pi$	$\arctan(-1)$	$-\pi/4$
$\arcsin(0)$	$0$	$\arccos(0)$	$\pi/2$	$\arctan(0)$	$0$
$\arcsin(1)$	$\pi/2$	$\arccos(1)$	$0$	$\arctan(1)$	$\pi/4$
$\arcsin(1/2)$	$\pi/6$	$\arccos(1/2)$	$\pi/3$	$\arctan(\sqrt{3})$	$\pi/3$
$\arcsin(-1/2)$	$-\pi/6$	$\arccos(-1/2)$	$2\pi/3$	$\arctan(-\sqrt{3})$	$-\pi/3$
$\arcsin(\sqrt{3}/2)$	$\pi/3$	$\arccos(\sqrt{3}/2)$	$\pi/6$	$\arctan(1/\sqrt{3})$	$\pi/6$
$\arcsin(-\sqrt{3}/2)$	$-\pi/3$	$\arccos(-\sqrt{3}/2)$	$5\pi/6$	$\arctan(-1/\sqrt{3})$	$-\pi/6$

2. (a) Dimostrare che

$$\arcsin x + \arccos x = \frac{\pi}{2} \quad \forall x \in [-1, 1].$$

(b) Dimostrare che

$$\arctan x + \arctan(1/x) = \begin{cases} \pi/2 & \text{se } x > 0, \\ -\pi/2 & \text{se } x < 0. \end{cases}$$

3. Determinare formule per le seguenti quantità (si raccomanda come sempre di quantificare)

$$\arcsin(-x) = \dots \quad \arccos(-x) = \dots \quad \arctan(-x) = \dots$$

4. Dimostrare la *formula di addizione* per l'arcotangente (prima di dimostrarla, occorre in realtà quantificarla per bene ...)

$$\arctan x + \arctan y = \arctan \left( \frac{x + y}{1 - xy} \right).$$

Dedurre la corrispondente formula di sottrazione.

5. Risolvere l'equazione

$$\arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5} + \arctan \frac{1}{x} = \frac{\pi}{4}.$$

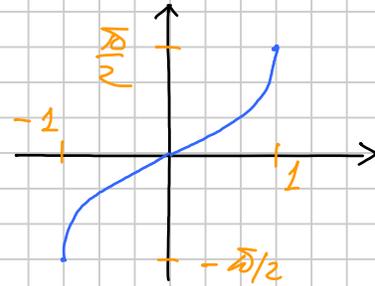
6. Semplificare le seguenti funzioni, scrivendole senza ricorrere a funzioni trigonometriche (occhio in tutti i casi a quantificare le formule):

$\sin(\arcsin x)$	$x \in [-1, 1]$	$\sin(\arccos x)$	$\sqrt{1-x^2}$ $-2 \leq x \leq 2$	$\sin(\arctan x)$	$\frac{x}{\sqrt{1+x^2}} \quad x \in \mathbb{R}$
$\cos(\arcsin x)$	$\sqrt{1-x^2}$ $-2 \leq x \leq 2$	$\cos(\arccos x)$	$x \in [-1, 1]$	$\cos(\arctan x)$	$\frac{1}{\sqrt{1+x^2}} \quad x \in \mathbb{R}$
$\tan(\arcsin x)$	$\frac{x}{\sqrt{1-x^2}} \quad  x  \leq 1$	$\tan(\arccos x)$	$\frac{\sqrt{1-x^2}}{x} \quad  x  \leq 1$	$\tan(\arctan x)$	$x \in [-1, 1]$

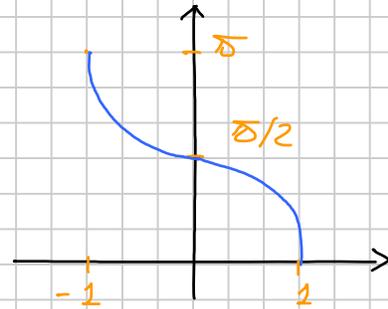
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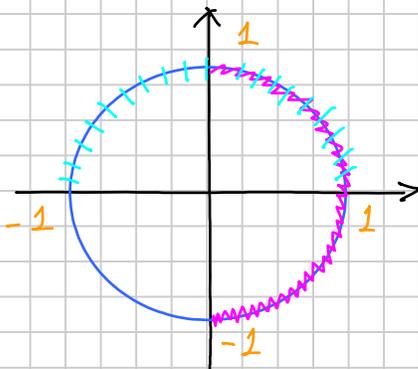
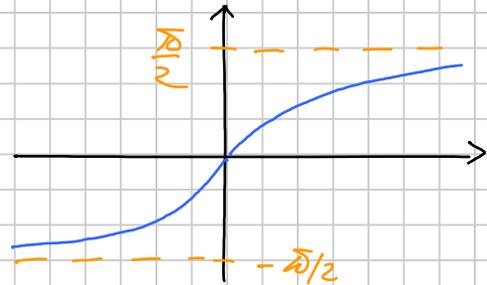
$\text{ARCSIN}(x) \quad [-1, 1] \rightarrow [-\pi/2, \pi/2]$



$\text{ARCCOS}(x) \quad [-1, 1] \rightarrow [0, \pi]$



$\text{ARCTAN}(x) \quad \mathbb{R} \rightarrow [-\pi/2, \pi/2]$



||||| ARCSIN    ARCTAN  
~~~~~ ARCCOS

2. (a) Dimostrare che

$$\arcsin x + \arccos x = \frac{\pi}{2} \quad \forall x \in [-1, 1].$$

$$y = \text{ARCSIN } x \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin y = x = \cos\left(\frac{\pi}{2} - y\right) \quad z = \frac{\pi}{2} - y \rightarrow z \in [0, \pi]$$

$$\rightarrow \text{ARCSIN}(\sin y) + \text{ARCCOS}(\cos(\frac{\pi}{2} - y)) = y + \frac{\pi}{2} - y = \frac{\pi}{2}$$

(b) Dimostrare che

$$\arctan x + \arctan(1/x) = \begin{cases} \pi/2 & \text{se } x > 0, \\ -\pi/2 & \text{se } x < 0. \end{cases}$$

$$y = \text{ARCTAN } x \quad x = \text{TAN } y \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

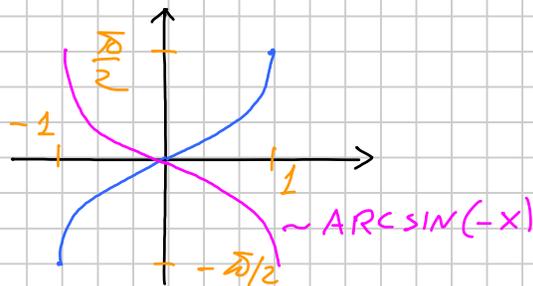
$$\frac{1}{x} = \text{COTAN } y = \begin{cases} \text{TAN}\left(\frac{\pi}{2} - y\right) & y \in (0, \frac{\pi}{2}) \quad x > 0 \\ \text{TAN}\left(-\frac{\pi}{2} - y\right) & y \in \left(-\frac{\pi}{2}, 0\right) \quad x < 0 \end{cases}$$

$$\text{ARCTAN}(x) + \text{ARCTAN}(1/x) = \begin{cases} y + \frac{\pi}{2} - y = \frac{\pi}{2} & x > 0 \\ y - \frac{\pi}{2} - y = -\frac{\pi}{2} & x < 0 \end{cases}$$

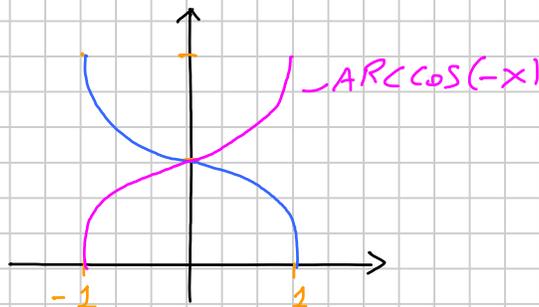
3. Determinare formule per le seguenti quantità (si raccomanda come sempre di quantificare)

$$\arcsin(-x) = \dots \quad \arccos(-x) = \dots \quad \arctan(-x) = \dots$$

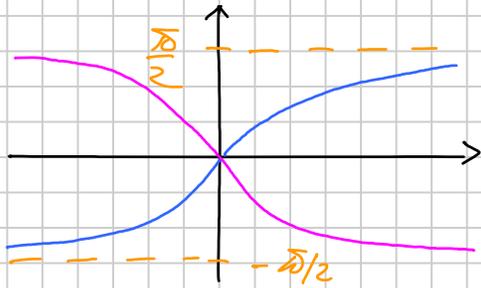
$$3.a) \quad \text{ARCSIN}(-x) = -\text{ARCSIN}(x) \quad \forall x \in [-1, 1]$$



$$3.b) \quad \text{ARCCOS}(-x) = \pi - \text{ARCCOS}(x) \quad \forall x \in [-1, 1]$$



$$3. c) \text{ARCTAN}(-x) = -\text{ARCTAN}(x) \quad \forall x \in \mathbb{R}$$



4. Dimostrare la formula di addizione per l'arcotangente (prima di dimostrarla, occorre in realtà quantificarla per bene ...)

$$\arctan x + \arctan y = \arctan \left( \frac{x+y}{1-xy} \right).$$

$$\text{ARCTAN } x + \text{ARCTAN } y = \text{ARCTAN} \left( \frac{x+y}{1-xy} \right)$$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R} \quad x \cdot y \neq 1 \quad -\frac{\pi}{2} < \text{ARCTAN } x + \text{ARCTAN } y < \frac{\pi}{2}$$

$$\begin{aligned} \text{TAN}(\text{ARCTAN } x + \text{ARCTAN } y) &= \frac{\text{TAN}(\text{ARCTAN } x) + \text{TAN}(\text{ARCTAN } y)}{1 - \text{TAN}(\text{ARCTAN } x) \cdot \text{TAN}(\text{ARCTAN } y)} = \\ &= \frac{x+y}{1-xy} = \text{TAN} \left( \text{ARCTAN} \left( \frac{x+y}{1-xy} \right) \right) \end{aligned}$$

Dedurre la corrispondente formula di sottrazione.

$$\text{ARCTAN } x - \text{ARCTAN } y = \text{ARCTAN } x + \text{ARCTAN}(-y) = \text{ARCTAN} \left( \frac{x-y}{1+xy} \right)$$

$$x \cdot y \neq -1 \quad -\frac{\pi}{2} < \text{ARCTAN } x - \text{ARCTAN } y < \frac{\pi}{2}$$

5. Risolvere l'equazione

$$\arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5} + \arctan \frac{1}{x} = \frac{\pi}{4}$$

$$\text{ARCTAN} \frac{1}{3} + \text{ARCTAN} \frac{1}{5} = \text{ARCTAN} \left( \frac{1/3 + 1/5}{1 - 1/3 \cdot 1/5} \right) =$$

$$= \text{ARCTAN} \left( \frac{7/15}{11/15} \right) = \text{ARCTAN} \left( \frac{7}{11} \right)$$

$$\begin{aligned} \operatorname{ARCTAN} \frac{7}{11} + \operatorname{ARCTAN} \frac{1}{5} &= \operatorname{ARCTAN} \left( \frac{7/11 + 1/5}{1 - 7/11 \cdot 1/5} \right) = \\ &= \operatorname{ARCTAN} \left( \frac{56/55}{58/55} \right) = \operatorname{ARCTAN} \left( \frac{23}{29} \right) \end{aligned}$$

$$\operatorname{ARCTAN} \left( \frac{23}{29} \right) + \operatorname{ARCTAN} \left( \frac{1}{x} \right) = \frac{\pi}{5} = \operatorname{ARCTAN} (1)$$

$$\operatorname{ARCTAN} \left( \frac{1}{x} \right) = \operatorname{ARCTAN} (1) - \operatorname{ARCTAN} \left( \frac{23}{29} \right) =$$

$$= \operatorname{ARCTAN} \left( \frac{1 - 23/29}{1 + 23/29} \right) = \operatorname{ARCTAN} \left( \frac{1/29}{57/29} \right) = \operatorname{ARCTAN} \left( \frac{1}{57} \right)$$

$$\leadsto x = 57$$

6. Semplificare le seguenti funzioni, scrivendole senza ricorrere a funzioni trigonometriche (occhio in tutti i casi a quantificare le formule):

|    |                   |                                           |                   |                                           |                   |                                                 |
|----|-------------------|-------------------------------------------|-------------------|-------------------------------------------|-------------------|-------------------------------------------------|
| 1) | $\sin(\arcsin x)$ | $x \in [-1, 1]$                           | $\sin(\arccos x)$ | $\frac{\sqrt{1-x^2}}{-2 \leq x \leq 1}$   | $\sin(\arctan x)$ | $\frac{x}{\sqrt{1+x^2}} \quad x \in \mathbb{R}$ |
| 2) | $\cos(\arcsin x)$ | $\frac{\sqrt{1-x^2}}{-2 \leq x \leq 1}$   | $\cos(\arccos x)$ | $x \in [-1, 1]$                           | $\cos(\arctan x)$ | $\frac{1}{\sqrt{1+x^2}} \quad x \in \mathbb{R}$ |
| 3) | $\tan(\arcsin x)$ | $\frac{x}{\sqrt{1-x^2}} \quad  x  \leq 1$ | $\tan(\arccos x)$ | $\frac{\sqrt{1-x^2}}{x} \quad  x  \leq 1$ | $\tan(\arctan x)$ | $x \in [-1, 1]$                                 |

$$1) \quad \sin(\operatorname{ARCSIN} x) = x \quad -1 \leq x \leq 1$$

$$\sin(\operatorname{ARCCOS} x) = \sqrt{1-x^2} \quad -2 \leq x \leq 1$$

$$\sin^2 \alpha = \sqrt{1 - \cos^2 \alpha}$$

$$\sin(\operatorname{ARCTAN} x) = \frac{x}{\sqrt{1+x^2}} \quad \forall x \in \mathbb{R}$$

$$\sin^2 \alpha = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha}$$

$$2) \quad \cos(\operatorname{ARCSIN} x) = \sqrt{1-x^2} \quad -1 \leq x \leq 1$$

$$\cos(\operatorname{ARCCOS} x) = x \quad -2 \leq x \leq 1$$

$$\cos(\operatorname{ARCTAN} x) = \frac{1}{\sqrt{1+x^2}} \quad \forall x \in \mathbb{R}$$

$$\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha}$$

$$3) \quad \text{TAN}(\text{ARCSIN} X) = \frac{X}{\sqrt{1-X^2}} \quad -1 \leq X \leq 1$$

$$\text{TAN} \alpha = \frac{\sin \alpha}{\sqrt{1-\sin^2 \alpha}}$$

$$\text{TAN}(\text{ARCCOS} X) = \frac{\sqrt{1-X^2}}{X} \quad -1 \leq X \leq 1$$

$$\text{TAN} \alpha = \frac{\sqrt{1-\cos^2 \alpha}}{\cos \alpha}$$

$$\text{TAN}(\text{ARCTAN} X) = X \quad X \in \mathbb{R}$$