BY GIMUSI 01/01/2015

Funzioni trigonometriche inverse 1

Argomenti: principio di induzione **Difficoltà:** ⋆⋆⋆

Prerequisiti: principio di induzione

1. Completare la seguente tabella

$\arcsin(-1)$	$-\pi/2$	$\arccos(-1)$	π	$\arctan(-1)$	$-\pi/4$
$\arcsin(0)$	0	$\arccos(0)$	$\pi/2$	$\arctan(0)$	0
$\arcsin(1)$	$\pi/2$	$\arccos(1)$	0	$\arctan(1)$	$\pi/4$
$\arcsin(1/2)$	$\pi/6$	$\arccos(1/2)$	$\pi/3$	$\arctan(\sqrt{3})$	$\pi/3$
$\arcsin(-1/2)$	$-\pi/6$	$\arccos(-1/2)$	$2\pi/3$	$\arctan(-\sqrt{3})$	$-\pi/3$
$\arcsin(\sqrt{3}/2)$	$\pi/3$	$\arccos(\sqrt{3}/2)$	$\pi/6$	$\arctan(1/\sqrt{3})$	$\pi/6$
$\arcsin(-\sqrt{3}/2)$	$-\pi/3$	$\arccos(-\sqrt{3}/2)$	$5\pi/6$	$\arctan(-1/\sqrt{3})$	$-\pi/6$

2. (a) Dimostrare che
$$\arcsin x + \arccos x = \frac{\pi}{2} \qquad \forall x \in [-1,1].$$
(b) Dimostrare che
$$\arctan x + \arctan(1/x) = \begin{cases} \pi/2 & \text{se } x > 0, \\ -\pi/2 & \text{se } x < 0. \end{cases}$$

3. Determinare formule per le seguenti quantità (si raccomanda come sempre di quantificare)
$\arcsin(-x) = \dots\dots\dots$ $\arccos(-x) = \dots\dots\dots$ $\arctan(-x) = \dots\dots\dots$

4. Dimostrare la *formula di addizione* per l'arcotangente (prima di dimostrarla, occorre in realtà quantificarla per bene ...)
$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right).$$
Dedurne la corrispondente formula di sottrazione.

5. Risolvere l'equazione
$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{x} = \frac{\pi}{4}.$$

6. Semplificare le seguenti funzioni, scrivendole senza ricorrere a funzioni trigonometriche (occhio in tutti i casi a quantificare le formule):

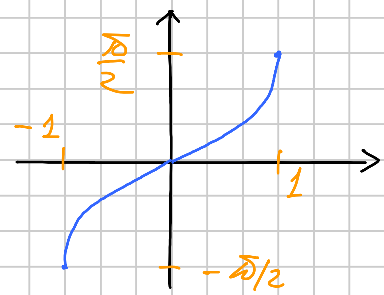
$\sin(\arcsin x)$	$x \in [-1,1]$	$\sin(\arccos x)$	$\sqrt{1-x^2}$ $-1 \le x \le 1$	$\sin(\arctan x)$	$\frac{x}{\sqrt{1+x^2}}$ $x \in \mathbb{R}$
$\cos(\arcsin x)$	$\sqrt{1-x^2}$ $-1 \le x \le 1$	$\cos(\arccos x)$	$x \in [-1,1]$	$\cos(\arctan x)$	$\frac{1}{\sqrt{1+x^2}}$ $x \in \mathbb{R}$
$\tan(\arcsin x)$	$\frac{x}{\sqrt{1-x^2}}$ $\lvert x \rvert \le 1$	$\tan(\arccos x)$	$\frac{\sqrt{1-x^2}}{x}$ $\lvert x \rvert \le 1$	$\tan(\arctan x)$	$x \in [-1,1]$

1/6

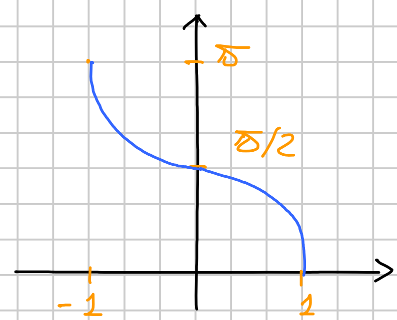
1. Completare la seguente tabella

$\arcsin(-1)$	$-\pi/2$	$\arccos(-1)$	π	$\arctan(-1)$	$-\pi/4$
$\arcsin(0)$	0	$\arccos(0)$	$\pi/2$	$\arctan(0)$	0
$\arcsin(1)$	$\pi/2$	$\arccos(1)$	0	$\arctan(1)$	$\pi/4$
$\arcsin(1/2)$	$\pi/6$	$\arccos(1/2)$	$\pi/3$	$\arctan(\sqrt{3})$	$\pi/3$
$\arcsin(-1/2)$	$-\pi/6$	$\arccos(-1/2)$	$2\pi/3$	$\arctan(-\sqrt{3})$	$-\pi/3$
$\arcsin(\sqrt{3}/2)$	$\pi/3$	$\arccos(\sqrt{3}/2)$	$\pi/6$	$\arctan(1/\sqrt{3})$	$\pi/6$
$\arcsin(-\sqrt{3}/2)$	$-\pi/3$	$\arccos(-\sqrt{3}/2)$	$5\pi/6$	$\arctan(-1/\sqrt{3})$	$-\pi/6$

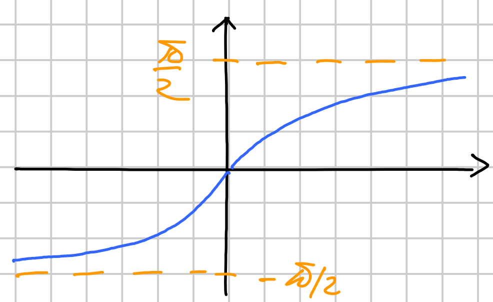
ARCSIN(x) $[-1, 1] \to [-\pi/2, \pi/2]$

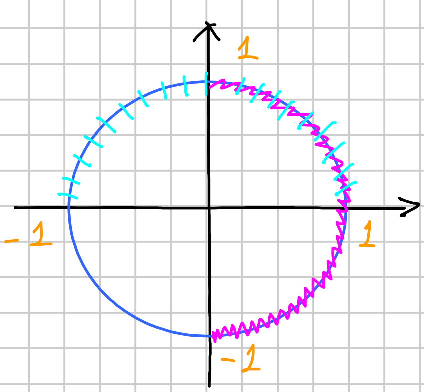


ARCCOS(x) $[-1, 1] \to [0, \pi]$



ARCTAN(x) $\mathbb{R} \to [-\pi/2, \pi/2]$





ARCSIN ARCTAN

ARCCOS

2. (a) Dimostrare che

$$\arcsin x + \arccos x = \frac{\pi}{2} \qquad \forall x \in [-1,1].$$

$Y = \text{ARCSIN}\, X \qquad Y \in [-\pi/2, \pi/2]$

$\sin Y = X = \cos\left(\frac{\pi}{2} - Y\right) \qquad Z = \frac{\pi}{2} - Y \rightsquigarrow Z \in [0, \pi]$

$\rightsquigarrow \quad \text{ARCSIN}(\sin Y) + \text{ARCCOS}(\cos(\pi/2 - Y)) = Y + \frac{\pi}{2} - Y = \frac{\pi}{2}$

(b) Dimostrare che

$$\arctan x + \arctan(1/x) = \begin{cases} \pi/2 & \text{se } x > 0, \\ -\pi/2 & \text{se } x < 0. \end{cases}$$

$Y = \text{ARCTAN}\, X \qquad X = \tan Y \qquad Y \in (-\pi/2, \pi/2)$

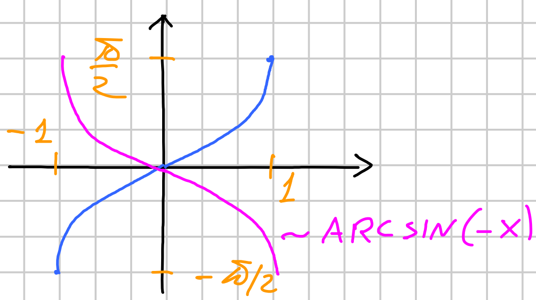
$$\frac{1}{X} = \text{COTAN}\, Y = \begin{cases} \tan\left(\frac{\pi}{2} - Y\right) & Y \in (0, \pi/2) \quad X > 0 \\ \tan\left(-\frac{\pi}{2} - Y\right) & Y \in \left(-\frac{\pi}{2}, 0\right) \quad X < 0 \end{cases}$$

$$\text{ARCTAN}(X) + \text{ARCTAN}(1/X) = \begin{cases} Y + \frac{\pi}{2} - Y = \frac{\pi}{2} & X > 0 \\ Y - \frac{\pi}{2} - Y = -\frac{\pi}{2} & X < 0 \end{cases}$$

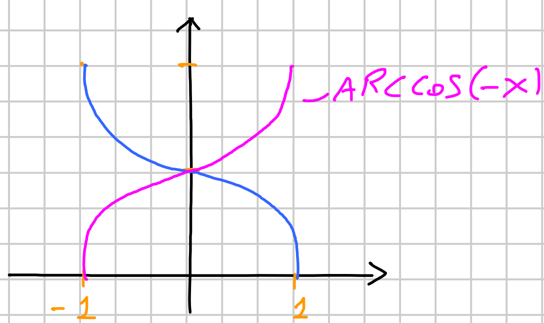
3. Determinare formule per le seguenti quantità (si raccomanda come sempre di quantificare)

$\arcsin(-x) = \ldots\ldots\ldots \qquad \arccos(-x) = \ldots\ldots\ldots \qquad \arctan(-x) = \ldots\ldots\ldots$

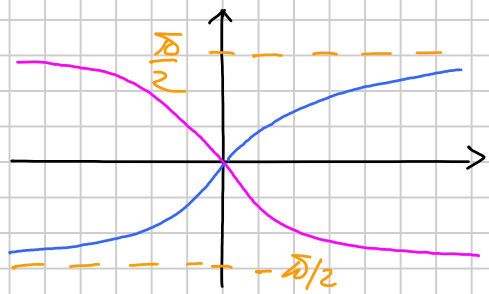
3.a) $\text{ARCSIN}(-X) = -\text{ARCSIN}(X) \qquad \forall X \in [-1, 1]$



3.b) $\text{ARCCOS}(-X) = \pi - \text{ARCCOS}(X) \qquad \forall X \in [-1, 1]$



3.C) $\arctan(-x) = -\arctan(x) \quad \forall x \in \mathbb{R}$



4. Dimostrare la *formula di addizione* per l'arcotangente (prima di dimostrarla, occorre in realtà quantificarla per bene ...)

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right).$$

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R} \quad x \cdot y \neq 1 \quad -\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}$$

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \cdot \tan(\arctan y)} =$$

$$= \frac{x+y}{1-xy} = \tan\left(\arctan\left(\frac{x+y}{1-xy}\right)\right)$$

Dedurne la corrispondente formula di sottrazione.

$$\arctan x - \arctan y = \arctan x + \arctan(-y) = \arctan\left(\frac{x-y}{1+xy}\right)$$

$$x \cdot y \neq -1 \quad -\frac{\pi}{2} < \arctan x - \arctan y < \frac{\pi}{2}$$

5. Risolvere l'equazione

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{x} = \frac{\pi}{4}.$$

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} = \arctan\left(\frac{1/3 + 1/4}{1 - 1/3 \cdot 1/4}\right) =$$

$$= \arctan\left(\frac{7/12}{11/12}\right) = \arctan(7/11)$$

$$\arctan\frac{7}{11}+\arctan\frac{1}{5}=\arctan\left(\frac{7/11+1/5}{1-7/11\cdot 1/5}\right)=$$

$$=\arctan\left(\frac{56/55}{58/55}\right)=\arctan(23/25)$$

$$\arctan\left(\frac{23}{25}\right)+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{4}=\arctan(1)$$

$$\arctan\left(\frac{1}{x}\right)=\arctan(1)-\arctan\left(\frac{23}{25}\right)=$$

$$=\arctan\left(\frac{1-23/25}{1+23/25}\right)=\arctan\left(\frac{1/25}{57/25}\right)=\arctan\left(\frac{1}{57}\right)$$

$$\rightsquigarrow x=57$$

6. Semplificare le seguenti funzioni, scrivendole senza ricorrere a funzioni trigonometriche (occhio in tutti i casi a quantificare le formule):

1)	$\sin(\arcsin x)$	$x\in[-1,1]$	$\sin(\arccos x)$	$\sqrt{1-x^2}$ $-1\le x\le 1$	$\sin(\arctan x)$	$\frac{x}{\sqrt{1+x^2}}$ $x\in\mathbb{R}$
2)	$\cos(\arcsin x)$	$\sqrt{1-x^2}$ $-1\le x\le 1$	$\cos(\arccos x)$	$x\in[-1,1]$	$\cos(\arctan x)$	$\frac{1}{\sqrt{1+x^2}}$ $x\in\mathbb{R}$
3)	$\tan(\arcsin x)$	$\frac{x}{\sqrt{1-x^2}}$ $\lvert x\rvert\le 1$	$\tan(\arccos x)$	$\frac{\sqrt{1-x^2}}{x}$ $\lvert x\rvert\le 1$	$\tan(\arctan x)$	$x\in[-1,1]$

1) $\sin(\arcsin x)=x \qquad -1\le x\le 1$

$\sin(\arccos x)=\sqrt{1-x^2} \qquad -1\le x\le 1 \qquad\qquad \sin\alpha=\pm\sqrt{1-\cos^2\alpha}$

$\sin(\arctan x)=\dfrac{x}{\sqrt{1+x^2}} \qquad \forall x\in\mathbb{R} \qquad\qquad \sin^2\alpha=\dfrac{\tan^2\alpha}{1+\tan^2\alpha}$

2) $\cos(\arcsin x)=\sqrt{1-x^2} \qquad -1\le x\le 1$

$\cos(\arccos x)=x \qquad -1\le x\le 1$

$\cos(\arctan x)=\dfrac{1}{\sqrt{1+x^2}} \qquad \forall x\in\mathbb{R} \qquad\qquad \cos^2\alpha=\dfrac{1}{1+\tan^2\alpha}$

3) $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}} \qquad -1 \leq x \leq 1$ $\qquad \tan\alpha = \dfrac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$

$\tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x} \qquad -1 \leq x \leq 1$ $\qquad \tan\alpha = \dfrac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$

$\tan(\arctan x) = x \qquad x \in \mathbb{R}$