$\int_A y\, dA \qquad A: \; x^2+y^2-y \le \sqrt{x^2+y^2}$

$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \qquad A: \; \rho^2-\rho\sin\theta \le \overset{(\rho\ge 0)}{\rho} \rightsquigarrow \rho \le 1+\sin\theta$

$$\int_A \rho\sin\theta\, dA = \int_0^{2\pi}\int_0^{1+\sin\theta} \rho\sin\theta \cdot \rho\, d\rho\, d\theta = \int_0^{2\pi}\int_0^{1+\sin\theta} \rho^2\sin\theta\, d\rho\, d\theta =$$

$$= \int_0^{2\pi} \sin\theta \left[\frac{1}{3}\rho^3\right]_0^{1+\sin\theta} d\theta = \frac{1}{3}\int_0^{2\pi} \sin\theta(1+\sin\theta)^3 d\theta =$$

$$= \frac{1}{3}\underbrace{\cancel{\int_0^{2\pi} \sin\theta\, d\theta}}_{=0} + \frac{1}{\cancel{3}}\underbrace{\int_0^{2\pi} \cancel{3}\sin^2\theta\, d\theta}_{=\pi} + \frac{1}{\cancel{3}}\underbrace{\cancel{\int_0^{2\pi} 3\sin^3\theta\, d\theta}}_{=0} + \frac{1}{3}\underbrace{\int_0^{2\pi} \sin^4\theta\, d\theta}_{=\frac{1}{4}\pi}$$

$$\int \sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + c \rightsquigarrow \int_0^{2\pi} \sin^2\theta = \pi$$

$$\int \sin^4\theta\, d\theta = \int \sin^2\theta(1-\cos^2\theta)\, d\theta = \int \sin^2\theta\, d\theta - \int \sin^2\theta\cos^2\theta\, d\theta =$$

$$\int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{8}\int \sin^2 2\theta\, d(2\theta) = \frac{1}{8}\left(\frac{2\theta}{2} - \frac{1}{4}\sin(4\theta)\right) =$$

$$= \frac{\theta}{8} - \frac{1}{32}\sin 4\theta$$

$$\int \sin^4\theta\, d\theta = \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c \rightsquigarrow \int_0^{2\pi} \sin^4 d\theta = \frac{3}{4}\pi$$

$$\rightsquigarrow \int y\, dA = \pi + \frac{\pi}{4} = \frac{5}{4}\pi$$

$$\int_A \sqrt{x^2+y^2}\, dA \qquad A:\ x^2+y^2 \ge 1,\ x^2+y^2 \le 2y$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad A: \begin{cases} \rho^2 \ge 1 \\ \rho^2 \ge 2\rho\operatorname{sen}\theta \end{cases} \quad \begin{cases} \rho \ge 1 \\ \rho \ge 2\operatorname{sen}\theta \end{cases}$$

$$\rightsquigarrow\ 2\operatorname{sen}\theta \ge 1 \quad \operatorname{sen}\theta \ge \frac{1}{2} \quad \rightsquigarrow\ \frac{\pi}{6} \le \theta \le \frac{5}{6}\pi$$

$$\int_A \sqrt{x^2+y^2}\, dA = \int_A \rho\, dA = \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\int_1^{2\operatorname{sen}\theta} \rho^2\, d\rho\, d\theta = \frac{1}{3}\int_{\pi/6}^{5/6\pi} \left[\rho^3\right]_1^{2\operatorname{sen}\theta} d\theta =$$

$$= \frac{1}{3}\int_{\pi/6}^{5/6\pi} (8\operatorname{sen}^3\theta - 1)\, d\theta = \frac{8}{3}\underbrace{\int_{\pi/6}^{5/6\pi} \operatorname{sen}^3\theta\, d\theta}_{=2\sqrt{3}} - \underbrace{\frac{1}{3}\int_{\pi/6}^{5/6\pi} d\theta}_{=\frac{2}{9}\pi}$$

$$\int \operatorname{sen}^3\theta\, d\theta = -\int (1-\cos^2\theta)\, d(\cos\theta) = -\cos\theta + \frac{1}{3}\cos^3\theta + c$$

$$\frac{8}{3}\int_{\pi/6}^{5/6\pi} \operatorname{sen}^3\theta\, d\theta = \frac{8}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_{\pi/6}^{5/6\pi} = \frac{8}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{\sqrt{3}}{2} - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3\right) =$$

$$= \frac{8}{3}\left(\sqrt{3} - \frac{\cancel{2}}{\cancel{3}}\,\frac{\cancel{3}\sqrt{3}}{\cancel{8}_{\,4}}\right) = \frac{\cancel{8}^{\,2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}}\sqrt{3} = 2\sqrt{3}$$

$$\frac{1}{3}\int_{\pi/6}^{5/6\pi} d\theta = \frac{1}{3}\left(\frac{5}{6}\pi - \frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{2}{3}\pi = \frac{2}{9}\pi$$

$$\rightsquigarrow \int \sqrt{x^2+y^2}\, dA = 2\sqrt{3} - \frac{2}{9}\pi$$