Integrali 5

Argomenti: Tecniche di integrazione **Difficoltà**: ⋆⋆⋆

Prerequisiti: Integrazione delle funzioni razionali

Determinare una primitiva delle seguenti funzioni (e fare la verifica). b

a

	Funzione	Primitiva	Funzione	Primitiva
1	$\frac{1}{x+3}$	$\log\lvert x+3\rvert$	$\frac{1}{3x+1}$	$\log\sqrt[3]{\lvert 3x+1\rvert}$
2	$\frac{1}{6x-5}$	$\log\sqrt[6]{\lvert 6x-5}$	$\frac{x-5}{x+5}$	$x-10\log\lvert x+5\rvert$
3	$\frac{x^2-5}{x+5}$	$\frac{1}{2}x^2-5x+20\log\lvert x+5\rvert$	$\frac{1}{x^2+2x-3}$	$\frac{1}{4}\log\left\lvert\frac{x-1}{x+3}\right\rvert$
4	$\frac{x}{x^2+1}$	$\frac{1}{2}\log(x^2+1)$	$\frac{1}{3x^2+2x-1}$	$\frac{1}{4}\log\left\lvert\frac{3x-1}{x+1}\right\rvert$
5	$\frac{3x+1}{x^2+1}$	$3\log\sqrt{x^2+1}+\arctan(x)$	$\frac{1}{x^2+2x+2}$	$\arctan(x+1)$
6	$\frac{x}{x^2+2x+2}$	$\frac{1}{2}\log(x^2+2x+2)-\arctan(x+1)$	$\frac{5x+3}{x^2+2x+2}$	$\frac{1}{2}\log(x^2+2x+2)-2\arctan(x+1)$
7	$\frac{1}{x^2+2}$	$\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)$	$\frac{1}{x^2+4}$	$\frac{1}{2}\arctan\left(\frac{x}{2}\right)$
8	$\frac{1}{2x^2+1}$	$\frac{1}{\sqrt{2}}\arctan(\sqrt{2}-x)$	$\frac{1}{2x^2+3}$	$\frac{1}{\sqrt{6}}\arctan\left(\frac{\sqrt{2}x}{\sqrt{3}}\right)$
9	$\frac{1}{x^2-1}$	$\frac{1}{2}\log\left\lvert\frac{x-1}{x+1}\right\rvert$	$\frac{1}{x^2-4}$	$\frac{1}{4}\log\left\lvert\frac{x-2}{x+2}\right\rvert$
10	$\frac{x^2}{x^2-1}$	$x+\frac{1}{2}\log\left\lvert\frac{x-1}{x+1}\right\rvert$	$\frac{x^2}{x^2+1}$	$x+\arctan(x)$
11	$\frac{1}{x^2-2}$	$\frac{1}{2\sqrt{2}}\log\left\lvert\frac{x-\sqrt{2}}{x+\sqrt{2}}\right\rvert$	$\frac{1}{(x^2+1)^2}$	$\frac{1}{2}\arctan(x)+\frac{x}{2(x^2+1)}$
12	$\frac{x^5}{x^3+1}$	$\frac{1}{3}(x^3-\log\lvert 1+x^3\rvert$	$\frac{1}{(x^2-1)^2}$	$\frac{1}{4}\log\left\lvert\frac{x+1}{x-1}\right\rvert-\frac{x}{2(x^2-1)}$
13	$\frac{x^2}{x^4-1}$	$\frac{1}{2}\arctan x+\frac{1}{4}\log\left\lvert\frac{x-1}{x+1}\right\rvert$	$\frac{x^5}{x^4-1}$	$\frac{1}{2}x^2+\frac{1}{4}\log\frac{\lvert x^2-1\rvert}{x^2+1}$
14	$\frac{x^6-1}{x^5+x^3}$	$\frac{1}{2}x^2+\frac{1}{2x^2}+\log\frac{\lvert x\rvert}{(x^2+1)}$	$\frac{x}{(x+1)^3}$	$-\frac{2x+1}{2(x+1)^2}$
15	$\frac{1}{x^2+x+1}$	$\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$	$\frac{x}{x^2+x+1}$	$\frac{1}{2}\log(x^2+x+1)-\frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$
16	$\frac{x}{x^3+1}$	$\log\frac{(x^2-x+1)^{1/6}}{\lvert x+1\rvert^{1/3}}+\frac{1}{\sqrt{3}}\mathrm{arct.}\left(\frac{2x-1}{\sqrt{3}}\right)$	$\frac{1}{x^4+1}$	(*)

(*) $\frac{1}{4\sqrt{2}}\log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}x+1)+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}x-1)$

INTEGRALI 5 Test n. 40 A.M. 1 Parte A

1)a $\int \frac{1}{x+3}\,dx = \boxed{\log|x+3|}$ $\left[\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

1b) $\int \frac{1}{3x+1}\,dx = \frac{1}{3}\int \frac{3}{3x+1}\,dx = \frac{1}{3}\log|3x+1| = \boxed{\log\sqrt[3]{|3x+1|}}$

2a) $\int \frac{1}{6x-5}\,dx = \frac{1}{6}\int \frac{6}{6x-5}\,dx = \boxed{\log\sqrt[6]{|6x-5|}}$

2b $\int \frac{x-5}{x+5}\,dx = \int \frac{x-5+5-5}{x+5}\,dx = \int \frac{x+5}{x+5}\,dx - \int \frac{10}{x+5}\,dx$

$= \int 1\,dx - 10\int \frac{1}{x+5}\,dx = \boxed{x - 10\log|x+5|}$

3a) $\int \frac{x^2-5}{x+5}\,dx$

$$\begin{array}{rr|l} x^2 & -5 & x+5 \\ -x^2 & -5x & x-5 \quad \boxed{A(x)} \\ \hline // & -5x-5 & \\ & +5x+25 & \\ \hline & 20 & \\ & \boxed{R(x)} & \end{array}$$

$\frac{x^2-5}{x+5} = \frac{\overset{Q(x)}{(x+5)}\overset{A(x)}{(x-5)} + \overset{R(x)}{20}}{x+5} =$

$= \int \frac{x^2-5}{x+5}\,dx = \int (x-5)\,dx + 20\int \frac{1}{x+5}\,dx =$

$= \boxed{\frac{1}{2}x^2 - 5x + 20\log|x+5|}$

3b) $\int \frac{1}{x^2+2x-3}\,dx$ $\qquad x^2+2x-3 = (x-1)(x+3)$

$S = -2 \quad P = -3$

$x = 1$ e $x = -3$

$$\frac{1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{Ax+3A+Bx-B}{(x-1)(x+3)}$$

$$\begin{cases} A+B=0 & \text{coeff. di } x \\ 3A-B=1 & \text{coeff. ter. noto} \end{cases} \quad B = 3A-1 \quad A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\frac{1}{x^2+2x-3} = \frac{1/4}{x-1} + \frac{-1/4}{x+3}$$

$$\int \frac{1}{x^2+2x-3}\,dx = \frac{1}{4}\left(\int \frac{1}{x-1}\,dx - \int \frac{1}{x+3}\,dx\right) = \boxed{\frac{1}{4}\log\left|\frac{x-1}{x+3}\right|}$$

4a) $$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \boxed{\frac{1}{2}\log(x^2+1)}$$

4b) $\int \frac{1}{3x^2+2x-1}\,dx$ $\qquad 3x^2+2x-1=0 \qquad x_{1,2} \begin{cases} = -1 \\ = 1/3 \end{cases}$

$(x+1) = 0$

$(x-1/3) = 0 \quad (3x-1) = 0$

$$\int \frac{1}{(3x-1)(x+1)}\,dx$$

$$\frac{1}{(3x-1)(x+1)} = \frac{A}{(3x-1)} + \frac{B}{(x+1)} = \frac{Ax+A+3Bx-B}{(3x-1)(x+1)}$$

$A + 3B = 0$ coeff x $\quad 1 + B + 3B = 0 \quad B = -1/4$

$A - B = 1$ coeff ter. noto $\quad A = 1 + B \quad A = 3/4$

$$\int \frac{3/4}{3x-1}\,dx + \int \frac{-1/4}{x+1}\,dx = \frac{1}{4}\int \frac{3}{3x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{4}\left(\log|3x-1| - \log|x+1|\right) = \boxed{\frac{1}{4}\log\left|\frac{3x-1}{x+1}\right|}$$

5a) $$\int \frac{3x+1}{x^2+1}\,dx = \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= \frac{3}{2}\log(x^2+1) + \arctan(x) = \boxed{3\log\sqrt{x^2+1} + \arctan x}$$

5b) $$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{1+(x+1)^2}\,dx = \quad y = x+1, \quad dy = 1\,dx$$

$$= \int \frac{1}{1+y^2}\,dy = \arctan(y) = \boxed{\arctan(x+1)}$$

6a) $$\int \frac{x}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+2} =$$

$$= \frac{1}{2}\left(\int \frac{2x+2}{x^2+2x+2}\,dx - 2\int \frac{1}{1+(x+1)^2}\,dx\right) = \boxed{\frac{1}{2}\log(x^2+2x+2) - \arctan(x+1)}$$

6b) $$\int \frac{5x+3}{x^2+2x+2}\,dx = \int \frac{5x+5-2}{x^2+2x+2}\,dx = \int \frac{5x+5}{x^2+2x+2}\,dx - \int \frac{2}{x^2+2x+2}\,dx$$

$$= \frac{5}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - 2\int \frac{1}{1+(x+1)^2}\,dx = \boxed{\frac{1}{2}\log(x^2+2x+2) - 2\arctan(x+1)}$$

7)a $\displaystyle\int \frac{1}{x^2+2}\,dx = \frac{1}{2}\int \frac{1}{1+\frac{x^2}{2}}\,dx = \frac{1}{2}\int \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}\,dx \qquad y = \frac{x}{\sqrt{2}} \quad dy = \frac{1}{\sqrt{2}}\,dx$

$$\frac{1}{\sqrt{2}}\int \frac{1}{1+y^2}\,dy = \frac{1}{\sqrt{2}}\arctan y \qquad \boxed{= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)}$$

da cui in generale si ha:

$$\textcircled{1}\ \boxed{\int \frac{1}{a+x^2}\,dx = \frac{1}{\sqrt{a}}\arctan\frac{x}{\sqrt{a}} \qquad \forall a>0}$$

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7)b $\displaystyle\int \frac{1}{x^2+4}\,dx = \frac{1}{\sqrt{4}}\arctan\frac{x}{\sqrt{4}} = \boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right)}$

8a) $\displaystyle\int \frac{1}{2x^2+1}\,dx = \frac{1}{2}\overset{\textcircled{1}}{\int} \frac{1}{\frac{1}{2}+x^2} = \frac{1}{2}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\arctan\left(\frac{x}{\frac{1}{\sqrt{2}}}\right) =$

$$= \boxed{\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\cdot x\right)}$$

8b) $\displaystyle\int \frac{1}{2x^2+3}\,dx \overset{\textcircled{1}}{=} \frac{1}{2}\int \frac{1}{x^2+\frac{3}{2}}\,dx = \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}}}\arctan\frac{x}{\sqrt{\frac{3}{2}}}$

$$= \left(\frac{1}{2}\frac{\sqrt{2}}{\sqrt{3}}\right) = \boxed{\frac{1}{\sqrt{6}}\arctan\left(\frac{\sqrt{2}\,x}{\sqrt{3}}\right)}$$

9a) $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{1+x-x}{x^2-1}\,dx = \int \frac{\cancel{x+1}\ 1}{\cancel{(x+1)}(x-1)}\,dx - \frac{1}{2}\int \frac{2x}{x^2-1}$

$$= \int \frac{1}{(x-1)} - \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \boxed{\log|x-1| - \frac{1}{2}\log|x^2-1|}$$

$$* = \boxed{\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|}$$

$$= \log|x-1| - \frac{1}{2}\log|(x-1)(x+1)|$$

$$\text{" \quad "} \quad -\frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| = \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| = *$$

9b) $\int \frac{1}{x^2-4}\,dx \quad \int \frac{1}{(x-2)(x+2)}\,dx$

$$\frac{1}{(x-2)(x+2)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} = \frac{Ax+2A+Bx-2B}{(x-2)(x+2)}$$

$A+B=0$ coeff x $\qquad A=\frac{1}{4}$

$2A-2B=1$ term. noto $\qquad B=-\frac{1}{4}$

$$\frac{1}{4}\int \frac{1}{(x-2)}\,dx - \frac{1}{4}\int \frac{1}{(x+2)}\,dx = \boxed{\frac{1}{4}\log\left|\frac{x-2}{x+2}\right|}$$

10a) $\int \frac{x^2}{x^2-1}\,dx \quad \int \frac{x^2+1-1}{x^2-1}\,dx = \int \frac{x-1}{x^2-1}\,dx + \underbrace{\int \frac{1}{x^2-1}\,dx}_{\text{Vedi 9a}}$

$$= \int 1\,dx + \int \frac{1}{(x-1)(x+1)}\,dx = \boxed{x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|}$$

10b) $\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int \frac{x^2+1}{x^2+1}\,dx - \int \frac{1}{1+x^2}\,dx$

$$= \boxed{x + \arctan x}$$

11a) $\int \frac{1}{x^2-2}\,dx \quad \frac{1}{(x-\sqrt{2})(x+\sqrt{2})} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}} = \frac{Ax+\sqrt{2}A+Bx-\sqrt{2}B}{(x-\sqrt{2})(x+\sqrt{2})}$

$A+B=0$ coeff x $\qquad A=\frac{1}{2\sqrt{2}} \qquad B=-\frac{1}{2\sqrt{2}}$

$\sqrt{2}A-\sqrt{2}B=1$ term. noto

$$= \frac{1}{2\sqrt{2}}\int \frac{1}{x-\sqrt{2}}\,dx - \frac{1}{2\sqrt{2}}\int \frac{1}{x+\sqrt{2}}\,dx = \boxed{\frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|}$$

11b) $\int \frac{1}{(x^2+1)^2}\,dx$; $\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{d}{dx}\frac{Cx+D}{x^2+1} =$

$$= \frac{Ax+B}{x^2+1} + \frac{C(x^2+1)-2x(Cx+D)}{(x^2+1)^2} = \frac{Ax^3+Ax+Bx^2+B+Cx^2+C-2Cx^2-2Dx}{(x^2+1)^2}$$

$A=0$ coeff x^3

$B-C=0$ coeff x^2

$A-2D=0$ coeff x

$B+C=1$ termine noto

$A=0$

$B=1-B$

$C=1-B$

$D=0$

$B=\frac{1}{2}$

$C=\frac{1}{2}$

$$\frac{1}{(x^2+1)^2} = \frac{\frac{1}{2}}{x^2+1} + \frac{d}{dx}\frac{\frac{1}{2}x}{x^2+1} =$$

$$\int \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{d}{dx}\frac{x}{x^2+1}$$

$$= \boxed{\frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)}}$$

12a) $\int \frac{x^5}{x^3+1}\,dx$

$$\begin{array}{l|l} x^5 & x^3+1 \\ \hline -x^5 - x^2 & x^2 \\ \hline // \quad -x^2 & \end{array}$$

$$\frac{x^5}{x^3+1} = \frac{(x^3+1)\cdot x^2 - x^2}{x^3+1} = \frac{\cancel{(x^3+1)}\cdot x^2}{\cancel{(x^3+1)}} - \frac{x^2}{x^3+1}$$

$$\int \frac{x^5}{x^3+1}\,dx = \int x^2\,dx - \frac{1}{3}\int \frac{3x^2}{x^3+1}\,dx$$

$$= \frac{1}{3}x^3 - \frac{1}{3}\log|x^3+1| \quad = \boxed{\frac{1}{3}\left(x^3 - \log|1+x^3|\right)}$$

12b) $\int \frac{1}{(x^2-1)^2}\,dx$; $\frac{1}{(x^2-1)^2} = \frac{1}{(x-1)^2(x+1)^2}$

2Dx

$$\frac{1}{(x-1)^2(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{d}{dx}\frac{Cx+D}{(x-1)(x+1)} = x^2-1$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C(x^2-1)-2x(Cx+D)}{(x-1)^2(x+1)^2}$$

$A(x^2-1)(x+1) = Ax^3+Ax^2-Ax-A$

$$\frac{A(x-1)(x+1)^2+B(x+1)(x-1)^2+Cx^2-C-2Cx^2-2Dx}{(x-1)^2(x+1)^2}$$

$$\frac{Ax^3+Ax^2-Ax-A+Bx^3-Bx^2-Bx+B+Cx^2-C-2Cx^2-2Dx}{(x-1)^2(x+1)^2}$$

$$\begin{cases} 1^\circ & A+B=0 \quad \text{coeff } x^3 \\ 2^\circ & A-B-C=0 \quad \text{coeff } x^2 \\ 3^\circ & A-B-2D=0 \quad \text{coeff } x \\ 4^\circ & -A+B-C=1 \quad \text{termine noto} \end{cases}$$

$B=-A$] $2^\circ + 4 \Rightarrow C = -\frac{1}{2}$

dalla 2° $-2A = \frac{1}{2}$ $A = -\frac{1}{4} \Rightarrow B = \frac{1}{4}$

dalla terza $2D=0$ $D=0$

$A = -1/4$ $B = 1/4$ $C = -1/2$ $D = 0$

$$\frac{-1/4}{x-1} + \frac{1/4}{x+1} + \frac{d}{dx}\frac{-\frac{1}{2}x}{x^2-1}$$

$$\frac{1}{4}\int\frac{1}{x+1}dx - \frac{1}{4}\int\frac{1}{x-1}dx - \frac{1}{2}\int\frac{d}{dx}\frac{x}{x^2-1}$$

$$= \boxed{\frac{1}{4}\log\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)}}$$

13a) $\int \frac{x^2}{x^4-1} = \left] \frac{x^2}{(x^2+1)(x+1)(x-1)} = \frac{x^2}{(x^4-1)} = \right.$

$$= \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} =$$

Numeratore: $(Ax+B)(x^2-1)+C(x^2+1)(x-1)+D(x^2+1)(x+1) =$

$= Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + Cx - C + Dx^3 + Dx^2 + Dx + D$

$$\begin{cases} A+C+D=0 & (\text{coeff } x^3) \\ B-C+D=1 & (\text{coeff } x^2) \\ -A+C+D=0 & (\text{coeff } x) \\ -B-C+D=0 & (\text{coeff. ter. noto}) \end{cases}$$

$1^\circ - 3^\circ \quad 2A=0$

$B = 1+C-D \Rightarrow B = 1-2D$

$C = -D$

$-1+2D+D+D \quad 4D=1$

$A=0$

$B=\frac{1}{2}$

$C=-\frac{1}{4}$

$D=\frac{1}{4}$

$$\frac{x^2}{x^4-1} = \frac{\frac{1}{2}}{x^2+1} + \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1}$$

$$\int \frac{x^2}{x^4-1}\,dx = \frac{1}{2}\int \frac{1}{1+x^2}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx$$

$$= \boxed{\frac{1}{2}\arctan x + \frac{1}{4}\log\left|\frac{x-1}{x+1}\right|}$$

13b $\int \frac{x^5}{x^4-1}\,dx$

$$\begin{array}{rr|l} x^5 & & x^4-1 \\ \hline -x^5 & +x & x \\ \hline & +x & \end{array}$$

$$\frac{x^5}{x^4-1} = \frac{(x^4-1)\cdot x + x}{(x^4-1)} = x + \frac{x}{x^4-1}$$

$$\frac{x}{x^4-1} = \frac{x}{(x^2+1)(x+1)(x-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

i conti sono gli stessi dell'esercizio 13 a il sistema è

$$\begin{cases} A+C+D=0 & (x^3) \\ B-C+D=0 & (x^2) \\ -A+C+D=1 & (x) \\ -B-C+D=0 & (t.n) \end{cases} \qquad \begin{array}{l} 2A=-1 \\ 2B=0 \\ D=C \\ 2C=\frac{1}{2} \end{array} \qquad \begin{array}{l} A=-\frac{1}{2} \\ B=0 \\ D=\frac{1}{4} \\ C=\frac{1}{4} \end{array}$$

$$\frac{x}{x^4-1}=\frac{-\frac{1}{2}x}{x^2+1}+\frac{\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}$$

$$\int\frac{x^5}{x^4-1}dx=\int x\,dx-\frac{1}{2\cdot 2}\int\frac{2x}{x^2+1}+\frac{1}{4}\int\frac{1}{x+1}dx+\frac{1}{4}\int\frac{1}{x-1}dx$$

$$=\frac{x^2}{2}-\frac{1}{4}\log(x^2+1)+\frac{1}{4}\log|x+1|+\frac{1}{4}\log|x-1|$$

$$=\boxed{\frac{1}{2}x^2+\frac{1}{4}\log\frac{|x^2-1|}{(x^2+1)}}$$

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146) $\int\frac{x^6-1}{x^5+x^3}$ divisione:

$$\begin{array}{rr|l} x^6 & -1 & x^5+x^3 \\ \hline -x^6-x^4 & & x \\ \hline // -x^4 & -1 & \end{array}$$

$$\frac{x^6-1}{x^5+x^3}=\frac{(x^5+x^3)\cdot x+(-x^4-1)}{x^5+x^3}=x-\frac{x^4+1}{x^5+x^3}$$

$$x-\left(\frac{x^4+1}{x^3(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{d}{dx}\frac{Dx+E}{x^2}\right)$$

molteplicità 3

calcolo $\frac{d}{dx}=\frac{Dx^2-2Dx^2-2Ex}{x^4}=$

$$=\frac{\cancel{x}(-Dx-2E)}{x^{\cancel{4}\,3}}$$

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{-Dx-2E}{x^3}=\frac{Ax^2(x^2+1)+(Bx+C)x^3(-Dx-2E)(x^2+1)}{x^3(x^2+1)}=$$

numeratore: $Ax^4+Ax^2+Bx^4+Cx^3-Dx^3-Dx-2Ex^2-2E$

sistema

$$\begin{array}{lll} A+B=1 & \text{coeff } x^4 & A=-1 \\ C-D=0 & \text{coeff } x^3 & B=2 \\ A-2E=0 & \text{coeff } x^2 & C=0 \\ -D=0 & \text{coeff } x & D=0 \\ -2E=1 & \text{t.n} & E=-\frac{1}{2} \end{array}$$

$$=\frac{-1}{x}+\frac{2x}{x^2+1}+\frac{d}{dx}\frac{-\frac{1}{2}}{x^2};$$

$$\frac{x^6-1}{x^5+x^3}=x+\frac{1}{x}-\frac{2x}{x^2+1}+\frac{d}{dx}\frac{\frac{1}{2}}{x^2}$$

$$\int\frac{x^6-1}{x^5+x^3}dx=\int x\,dx+\int\frac{1}{x}dx-\int\frac{2x}{x^2+1}dx+\int\frac{d}{dx}\frac{1}{2x^2}$$

$$=\frac{x^2}{2}+\log|x|-\log(x^2+1)+\frac{1}{2x^2}$$

$$=\boxed{\frac{1}{2}x^2+\frac{1}{2x^2}+\log\frac{|x|}{(x^2+1)}}$$

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146) $\displaystyle\int\frac{x}{(x+1)^3}dx \qquad \frac{x}{(x+1)^3}=\frac{A}{(x+1)}+\frac{d}{dx}\frac{Bx+C}{(x+1)^2}$

$$=\frac{A}{(x+1)}+\frac{B(x+1)^{\cancel{2}}-2\cancel{(x+1)}(Bx+C)}{(x+1)^{\cancel{4}\,3}}=\frac{A}{x+1}+\frac{-Bx+B-2C}{(x+1)^3}$$

numeratore $Ax^2 + A + 2Ax - Bx + B - 2C$

$$\begin{cases} A = 0 & \text{coeff } x^2 \\ 2A - B = 1 & \text{coeff } x \\ A + B - 2C = 0 & \text{t.n} \end{cases}$$

$A = 0$; $B = -1$ $C = -\frac{1}{2}$

$$\frac{x}{(x+1)^3} = \frac{d}{dx} \frac{-x - 1/2}{(x+1)^2} \qquad \int \frac{x}{(x+1)^3}\, dx = -\int \frac{d}{dx} \frac{x + 1/2}{(x+1)^2}$$

$$= \boxed{-\frac{2x+1}{2(x+1)^2}}$$

15a) $\int \frac{1}{x^2 + x + 1}\, dx$

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \boxed{?}$$

$$= x^2 + \frac{1}{4} + x + \frac{3}{4} = \frac{3}{4} + \left(x + \frac{1}{2}\right)^2$$

$$\int \frac{1}{\frac{3}{4} + \left(x + \frac{1}{2}\right)^2}\, dx = \frac{1}{\sqrt{\frac{3}{4}}} \arctan\left(\frac{x + 1/2}{\sqrt{\frac{3}{4}}}\right) =$$

Vedere esercizio 7a

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2\left(x + \frac{1}{2}\right)}{\sqrt{3}}\right) = \boxed{\frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right)}$$

15b) $\int \frac{x}{x^2 + x + 1}\, dx$

$$\frac{x + 1/2 - 1/2}{x^2 + x + 1} = \frac{2}{2}\frac{(x + 1/2)}{x^2 + x + 1} + \frac{-\frac{1}{2}}{x^2 + x + 1}$$

$$= \frac{1}{2} \frac{2x+1}{x^2 + x + 1} - \frac{1}{2} \frac{1}{x^2 + x + 1} \; ; \; \frac{1}{2}\int \frac{2x+1}{x^2 + x + 1}\, dx - \frac{1}{2} \int \underbrace{\frac{1}{x^2 + x + 1}}_{\text{esercizio 15a}}$$

$$\int \frac{x}{x^2 + x + 1}\, dx = \boxed{\frac{1}{2} \log(x^2 + x + 1) - \frac{1}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right)}$$

16a) $\int \frac{x}{x^3+1}\,dx$

$$\frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{Ax^2-Ax+A+Bx^2+Bx+Cx+C}{(x+1)(x^2-x+1)}$$

$$\begin{array}{llll} A+B=0 & A=-B & & A=-\frac{1}{3} \\ -A+B+C=1 & B+B+C=1 & B+B+B=1 & B=\frac{1}{3} \\ A+C=0 & C=-A & & C=\frac{1}{3} \end{array}$$

1

$$= \boxed{\frac{-\frac{1}{3}}{x+1}} + \frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}$$

1 $-\frac{1}{3}\int \frac{1\,dx}{x+1} = -\frac{1}{3}\log|x+1|$

2

$$\frac{1}{3}\cdot\frac{2}{2}\left(\frac{x+1-\frac{1}{2}+\frac{1}{2}}{(x^2-x+1)}\right) = \frac{1}{6}\left(\boxed{\frac{2x-1}{x^2-x+1}} + \frac{3}{x^2-x+1}\right)$$

2 $\frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx = \frac{1}{6}\log(x^2-x+1)$

3

$$\frac{1}{2\cancel{6}}\left(\frac{\cancel{3}\,1}{x^2-x+1}\right) = \boxed{\frac{1}{2}\,\frac{1}{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}}$$

vedere 7a

3 $\frac{1}{2}\int \frac{1}{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}\,dx =$

$$= \frac{1}{2}\,\frac{1}{\sqrt{\frac{3}{4}}}\arctan\left(\frac{\left(x-\frac{1}{2}\right)}{\sqrt{\frac{3}{4}}}\right) = \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$\int \frac{x}{x^3+1}\,dx = \boxed{\log\left(\frac{(x^2-x+1)^{1/6}}{|x+1|^{1/3}}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}$$

16b) $\displaystyle\int \frac{1}{x^4+1}\,dx$ $\left[x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2 \quad (A^2-B^2) \right]$

$$\frac{1}{x^4+1} = \frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$= \frac{Ax^3-\sqrt{2}Ax^2+Ax+Bx^2-\sqrt{2}Bx+B+Cx^3+\sqrt{2}Cx^2+Cx+Dx^2+\sqrt{2}Dx+D}{(x^2+\sqrt{2}x-1)(x^2-\sqrt{2}x+1)}$$

sistema

$$\begin{array}{ll|l}
A+C=0 & \text{coeff. } (x^3) & A=-C \\
-\sqrt{2}A+B+\sqrt{2}C+D=0 & (x^2) & 2\sqrt{2}C+B+D=0 \quad B+D=-2\sqrt{2}C=1 \\
+A-\sqrt{2}B+C+\sqrt{2}D=0 & (x) & \sqrt{2}D=\sqrt{2}B \qquad C=-\frac{1}{2\sqrt{2}} \\
B+D=1 & (t.n) & D=B,\ D=\frac{1}{2},\ B=\frac{1}{2} \qquad A=-C=\frac{1}{2\sqrt{2}}
\end{array}$$

$$A=\frac{1}{2\sqrt{2}};\quad B=\frac{1}{2}\quad C=-\frac{1}{2\sqrt{2}},\quad D=\frac{1}{2}$$

a, b

$$= \boxed{\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}} + \boxed{\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}}$$

moltiplico e divido il numeratore per $4\sqrt{2}$ e spezzo la frazione in 2
1° parte si ricava un log(- - -)
2° parte si ricava una arct(- - -)

(a) $$\frac{1}{4\sqrt{2}}\,\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{1}{4\cancel{\sqrt{2}}}\,\frac{\cancel{\sqrt{2}}}{x^2+\sqrt{2}x+1}$$

1° parte

$$\frac{1}{4\sqrt{2}}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}+1}\,dx = \frac{1}{4\sqrt{2}}\log\left(x^2+\sqrt{2}+1\right)$$

2° parte

$$\frac{1}{4}\int \frac{1}{\frac{1}{2}+\left(x+\frac{\sqrt{2}}{2}\right)^2}\,dx = \frac{1}{4}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\arctan\left(\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right) = \frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)$$

Vedere esercizio 7a

Per quanto riguarda la frazione b i conteggi sono gli stessi cambiano solo alcuni segni.

$$\int \frac{1}{x^4+1}\,dx = \boxed{\frac{1}{4\sqrt{2}} \log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}} \arctan\left(\sqrt{2}x+1\right) + \frac{1}{2\sqrt{2}} \arctan\left(\sqrt{2}x-1\right)}$$

<u>Fine</u>