Integrali 4

Argomenti: Tecniche di integrazione **Difficoltà:** ⋆⋆⋆

Prerequisiti: Integrazione per parti e per sostituzione

Determinare una primitiva delle seguenti funzioni (e fare la verifica).

Funzione	Primitiva	Funzione	Primitiva
$\sin(x+3)$	$-\cos(x+3)$	$\cos(3x+2)$	$\frac{1}{3}\sin(3x+2)$
$\sqrt{x+3}$	$\frac{2}{3}(x+3)^{3/2}$	$\sqrt{2x+5}$	$\frac{1}{3}(2x+5)^{3/2}$
$\sinh(4x+5)$	$\frac{1}{4}\cosh(4x+5)$	$\tan x$	$-\log\lvert\cos x\rvert$
xe^{x^2}	$\frac{1}{2}e^{x^2}$	$x^3\cos(x^4)$	$\frac{1}{4}\sin x^4$
$\dfrac{\sin x}{\cos^2 x}$	$\dfrac{1}{\cos x}$	$\dfrac{\cos x}{\sin^3 x}$	$-\dfrac{1}{2\sin^2 x}$
$\dfrac{1}{\sqrt[3]{3x+2}}$	$\frac{1}{2}(3x+2)^{2/3}$	$\dfrac{1}{(x+1)^2}$	$-\dfrac{1}{x+1}$
$\dfrac{\log x}{x}$	$\frac{1}{2}\log^2 x$	$\dfrac{\log^7 x}{x}$	$\frac{1}{8}\log^8 x$
$\dfrac{1}{x\log^3 x}$	$-\dfrac{1}{2\log^2 x}$	$\dfrac{1}{x\log x}$	$\log\lvert\log x\rvert$
$\dfrac{x}{1+x^2}$	$\frac{1}{2}\log(1+x^2)$	$\dfrac{x}{1+x^4}$	$\frac{1}{2}\arctan(x^2)$
$\dfrac{x^3}{1+x^4}$	$\frac{1}{4}\log(1+x^4)$	$\dfrac{e^x}{1+e^{2x}}$	$\arctan(e^x)$
$x^3e^{-x^2}$	$-\frac{1}{2}(x^2+1)e^{-x^2}$	$x^5e^{2x^3}$	$\frac{1}{12}(2x^3-1)e^{2x^3}$
$\sin x\sqrt{\cos x}$	$-\frac{2}{3}(\cos x)^{3/2}$	$\tan(7x)$	$-\frac{1}{7}\log\lvert\cos(7x)\rvert$
$\log(2x-5)$	$\frac{1}{2}(2x-5)(\log(2x-5)-1)$	$\sqrt{x}e^{\sqrt{x}}$	$2e^{\sqrt{x}}(x-2\sqrt{x}+2)$
$\cos x\cdot e^{\sin x}$	$e^{\sin x}$	$\sin^3 x\cdot e^{\cos x}$	$e^{\cos x}(\cos x-1)^2$
$\cos^3 x\sin^4 x$	$\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x$	$\sin(\log x)$	$\frac{x}{2}(\sin(\log x)-\cos(\log x))$
$\dfrac{\tan(\sqrt{x})}{\sqrt{x}}$	$-2\log\lvert\cos\sqrt{x}\rvert$	$\dfrac{1}{\sin^3 x\cos^3 x}$	$\dfrac{1}{2\cos^2 x}-\dfrac{1}{2\sin^2 x}+2\log\lvert\tan x\rvert$

INTEGRALI 4 Esercizio A.M.1 Parte A

1a) $\int \sin(x+3)\,dx$ pongo $y = x+3$ $\quad \frac{dy}{dx} = 1 \quad dy = dx$

$\int \sin y \, dy = -\cos y$ torno in x $\boxed{-\cos(x+3)}$

1b) $\int \cos(3x+2)\,dx$ pongo $y = 3x+2$ $\quad \frac{dy}{dx} = 3$

$\frac{1}{3}\int \cos y \, dy = \frac{1}{3}\sin y \longrightarrow \boxed{\frac{1}{3}\sin(3x+2)}$

2a) $\int \sqrt{x+3}\,dx$ $\quad y = x+3 \quad \frac{dy}{dx} = 1$

$\int \sqrt{y}\,dy = \frac{2}{3}y^{3/2} \rightarrow \boxed{\frac{2}{3}(x+3)^{3/2}}$

2b) $\int \sqrt{2x+5}\,dx$ $\quad y = 2x+5 \quad \frac{dy}{dx} = 2$

$\frac{1}{2}\int \sqrt{y}\,dy = \frac{1}{2}\cdot\frac{2}{3}y^{3/2} \longrightarrow \boxed{\frac{1}{3}(2x+5)^{3/2}}$

3a) $\int \sinh(4x+5)\,dx$ $\quad y = 4x+5 \quad \frac{dy}{dx} = 4$

$\frac{1}{4}\int \sinh(y)\,dy = \frac{1}{4}\cosh(y) = \boxed{\frac{1}{4}\cosh(4x+5)}$

3b) $\int \tan x \, dx = \int \frac{\sin x}{\cos x}$ $\quad y = \cos x \quad \frac{dy}{dx} = -\sin x$

$= -\int \frac{1}{y} dy = -\log|y| \longrightarrow \; = -\boxed{\log|\cos x|}$

4a) $\int x e^{x^2} dx \quad y = e^{x^2} \quad \frac{dy}{dx} = 2x e^{x^2} \quad dy = 2x e^{x^2} dx$

$\frac{1}{2} \int dy = y/2 \Rightarrow \boxed{\frac{e^{x^2}}{2}}$

4b) $\int x^3 \cos(x^4) \, dx \quad y = x^4 \quad \frac{dy}{dx} = 4x^3 \quad dy = 4x^3 dx$

$\frac{1}{4} \int \cos(y) \, dy = \frac{1}{4} \sin(y) = \boxed{\frac{1}{4} \sin(x^4)}$

5a) $\int \frac{\sin x}{\cos^2 x} dx \quad y = \cos x \quad dy = -\sin x \, dx$

$-\int \frac{dy}{y^2} = -\left(-\frac{1}{y}\right) \Longrightarrow \boxed{= \frac{1}{\cos x}}$

5b) $\int \frac{\cos x}{\sin^3 x} dx \quad y = \sin x \quad dy = \cos x \, dx$

$\int \frac{dy}{y^3} = -\frac{1}{2y^2} \Longrightarrow \boxed{-\frac{1}{2 \sin^2 x}}$

6a) $\int \frac{1}{\sqrt[3]{3x+2}}\,dx \qquad y = 3x+2 \qquad dy = 3\,dx$

$$\frac{1}{3}\int \frac{dy}{\sqrt[3]{y}} = \cancel{\frac{1}{3}}\,\frac{\cancel{3}}{2}\,\cancel{y}^{2/3} \Longrightarrow \boxed{\frac{1}{2}(3x+2)^{2/3}}$$

6b) $\int \frac{1}{(x+1)^2}\,dx \qquad y = x+1 \qquad dy = 1\,dx$

$$\int \frac{dy}{y^2} = -\frac{1}{y} \Longrightarrow \boxed{-\frac{1}{x+1}}$$

7a) $\int \frac{\log x}{x}\,dx \qquad y = \log x \qquad dy = \frac{1}{x}\,dx$

$$\int y\,dy = \frac{y^2}{2} \Longrightarrow \boxed{\frac{\log^2 x}{2}}$$

7b) $\int \frac{\log^7 x}{x}\,dx \qquad y = \log x \qquad dy = \frac{1}{x}\,dx$

$$\int y^7 \cdot dy = \frac{1}{8}y^8 \Longrightarrow \boxed{\frac{1}{8}\log^8 x}$$

8a) $\int \frac{1}{x\log^3 x}\,dx \qquad y = \log x \qquad dy = \frac{1}{x}\,dx$

$$\int \frac{dy}{y^3} = -\frac{1}{2y^2} = \boxed{-\frac{1}{2\log^2 x}}$$

8b) $\int \frac{1}{x \log x} dx \qquad y = \log x \qquad dy = \frac{1}{x} dx$

$$\int \frac{dy}{y} = \log|y| = \boxed{\log|\log x|}$$

9a) $\frac{1}{2}\int \frac{2x}{1+x^2} dx \qquad dy = 1+x^2 = 2x\, dx \quad \text{con } y = 1+x^2$

$$= \frac{1}{2}\int \frac{dy}{y} = \frac{1}{2}\log|y| \Longrightarrow \boxed{\frac{1}{2}\log(1+x^2)}$$

9b) $\int \frac{x}{1+x^4} dx = \frac{1}{2}\int \frac{2\cdot 1}{1+(x^2)^2} dx \qquad y = x^2 \quad dy = 2x\,dx$

$$\frac{1}{2}\int \frac{1}{1+y^2} dy = \frac{1}{2}\arctan(y) = \boxed{\frac{1}{2}\arctan(x^2)}$$

10a) $\int \frac{x^3}{1+x^4} dx = \frac{1}{2}\int \frac{2x \cdot x^2}{1+(x^2)^2} dx \qquad y = x^2 \quad dy = 2x\,dx$

$$\frac{1}{2}\frac{1}{2}\int \frac{2y}{1+y^2} dy = \frac{1}{4}\int \frac{2y}{1+y^2} dy = \frac{1}{4}\log(1+y^2)$$

$$\boxed{= \frac{1}{4}\log(1+x^4)}$$

10b) $\int \frac{e^x}{1+e^{2x}} dx \qquad y = e^x \quad dy = e^x dx \qquad \int \frac{1}{1+y^2} dy =$

$$= \arctan(y) = \boxed{\arctan(e^x)}$$

11 a) $\int x^3 e^{-x^2}\,dx = \int x^2\, x e^{-x^2}\,dx \qquad y = x^2 \quad dy = 2x\,dx$

$\frac{1}{2}\int \underset{F}{y}\cdot \underset{g}{e^{-y}}\,dy$ per parti $= \frac{1}{2}\left[-\underset{F}{y}\,\underset{G}{e^{-y}} - \int \underset{f}{1}\,\underset{G}{(-e^{-y})}\,dy\right]$

$= \frac{1}{2}\left[-ye^{-y} - (e^{-y})\right] \Rightarrow \frac{1}{2}\left(-x^2e^{-x^2} - e^{-x^2}\right) = \boxed{-\frac{1}{2}(x^2+1)\,e^{-x^2}}$

11b) $\int x^5 e^{2x^3}\,dx = \int x^3\, x^2 e^{2x^3}\,dx \qquad y = x^3 \quad dy = 3x^2\,dx$

$\frac{1}{3}\int \underset{F}{y}\, \underset{g}{e^{2y}}\cdot dy = \frac{1}{3}\left[y\cdot\frac{1}{2}e^{2y} - \int 1\,\frac{1}{2}e^{2y}\,dy\right]$

$= \frac{1}{3}\left[\frac{1}{2}e^{2y}\cdot y - \frac{1}{2}\cdot\frac{1}{2}e^{2y}\right] = \frac{1}{6}x^3\cdot e^{2x^3} - \frac{1}{12}e^{2x^3} = \boxed{\frac{1}{12}(2x^3-1)\,e^{2x^3}}$

12a) $\int \sin x\sqrt{\cos x}\,dx \qquad y = \cos x \qquad dy = -\sin x\,dx$

$-\int\sqrt{y}\,dy = -\frac{2y^{3/2}}{3} = \boxed{-\frac{2}{3}(\cos x)^{3/2}}$

12b $\int \tan(7x)\,dx = \int \frac{\sin 7x}{\cos 7x}\,dx \qquad y = \cos 7x \qquad dy = -7\sin 7x\,dx$

$-\frac{1}{7}\int\frac{1}{y}\,dy = -\frac{1}{7}\log|y| = \boxed{-\frac{1}{7}\log|\cos 7x|}$

13a) $\int \log(2x-5)\,dx \qquad y = 2x-5 \qquad dy = 2\,dx$

$$\frac{1}{2}\int \log(y)\,dy = \frac{1}{2}\int \log y \cdot 1\,dy = \log(y)\cdot y - \int \frac{1}{y}\, y\,dy$$

$$= \frac{1}{2}\left(y\log(y) - y\right) = \frac{1}{2}y\left(\log(y) - 1\right) \Rightarrow \text{tornare in } x$$

$$= \boxed{\frac{1}{2}(2x-5)\left(\log(2x-5) - 1\right)}$$

13b) $\int \sqrt{x}\, e^{\sqrt{x}}\,dx \qquad t = \sqrt{x} \qquad dt = \frac{1}{2\sqrt{x}}\,dx \qquad dt = \frac{1}{2t}\,dx \qquad 2t\,dt = dx$

Sostituzione e 2 volte per parti

$$2\int t^2 e^t\,dt = 2\left[t^2 e^t \boxed{- 2\int t e^t\,dx}\right] =$$

$$\boxed{-2\left[t\cdot e^t - \int 1 e^t\,dt\right] = -2\left(t\cdot e^t - e^t\right)}$$

$$2\int t^2 e^t\,dt = 2\left[t^2 e^t - 2\left(t e^t - e^t\right)\right] = 2e^t\left(t^2 - 2t + 2\right) \Rightarrow \text{tornare in } x$$

$$= \boxed{2e^{\sqrt{x}}\left(x - 2\sqrt{x} + 2\right)}$$

14a) $\int \cos x \cdot e^{\sin x}\,dx \qquad e^{\sin x} = y \qquad dy = \cos x\, e^{\sin x}\,dx$

$$\int dy = \int 1\,dy = y \Longrightarrow \boxed{e^{\sin x}}$$

14b) $\int \sin^3 x \cdot e^{\cos x}\, dx = \int \sin^2 x \cdot \sin x\; e^{\cos x}\, dx$; $t = \cos x$; $dt = -\sin x\, dx$

$-\int (1-\cos^2 x)\, e^t\, dt = -\int (1-t^2)\, e^t\, dt = -\int e^t dt + \underbrace{\int t^2 e^t\, dt}_{\text{già fatto}}$ 13b

$= -e^t + e^t(t^2-2t+2) = e^t(t^2-2t+1)$

$= e^{\cos x}(\cos^2 x - 2\cos x + 1) = \boxed{e^{\cos x}(\cos x - 1)^2}$

15a) $\int \cos^3 x \sin^4 x\, dx = \int \cos^2 x \cdot \cos x \sin^4 x\, dx$

$t = \sin x \quad dt = \cos x\, dx \quad = \int (1-\sin^2 x)\cdot t^4\, dt$

$= \int (1-t^2)\cdot t^4\, dt = \int t^4 dt - \int t^6 dt = \frac{t^5}{5} - \frac{t^7}{7}$

$= \boxed{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x}$

15b) $\int \sin(\log x)\, dx$ 2 volte x parti con 1 nascosto + grande ritorno.

$\int \underset{g}{1}\, \underset{F}{\sin(\log x)}\, dx = x\sin(\log x) - \boxed{\int \cos(\log x)\cdot \frac{1}{\cancel{x}}\cdot \cancel{x}\, dx}$

$\boxed{-\int \cos(\log x)\cdot 1\, dx = -\left(x\cos(\log x) - \int -\sin\log(x)\,\frac{1}{\cancel{x}}\cdot\cancel{x}\, dx\right.}$

$\int \sin(\log x)\, dx = x\sin(\log x) - x\cos(\log x) - \underbrace{\int \sin(\log x)\, dx}_{\text{Grande Ritorno}}$

dx $\quad 2\int \sin(\log x)\,dx = x\left(\sin(\log x) - \cos(\log x)\right)$

$$\boxed{= \frac{x}{2}\left(\sin(\log x) - \cos(\log x)\right)}$$

16a) $\displaystyle\int \frac{\tan(\sqrt{x})}{\sqrt{x}}\,dx \qquad y = \cos(\sqrt{x}) \qquad dy = -\sin\sqrt{x}\,\frac{1}{2\sqrt{x}}\,dx$

$$\int \frac{\sin\sqrt{x}}{\cos\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\,dx = -2\int \frac{1}{y}\,dy = -2\log|y|$$

$$= \boxed{-2\log|\cos\sqrt{x}|}$$

16b) $\displaystyle\int \frac{1}{\sin^3 x \cdot \cos^3 x}\,dx$ $\qquad$ Vedere lezione 81 del 23/11/2011

$$= \int \frac{(\cos^2 x + \sin^2 x)^2}{\sin^3 x \cos^3 x}\,dx = \int \frac{\overset{1}{\cos^4 x} + \overset{2}{2\cos^2\sin^2 x} + \overset{3}{\sin^4 x}}{\sin^3 x \cos^3 x}\,dx$$

1 $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx \qquad y = \sin x \quad dy = \cos x\,dx \qquad \int \frac{dy}{y^3} = -\frac{1}{2y^2} \Rightarrow \; = \boxed{-\frac{1}{2\sin^2 x}}$

2 (*) $\displaystyle 2\int \frac{1}{\sin x \cos x}\,dx = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\,dx = \int \overset{2a}{\frac{\sin x}{\cos x}}\,dx + \int \overset{2b}{\frac{\cos x}{\sin x}}\,dx$

* 2a $\quad y = \cos x \quad dy = -\sin x\,dx \qquad \displaystyle -\int \frac{1}{y}\,dy \Rightarrow -\log|\cos x|$

2b $\quad y = \sin x \quad dy = \cos x\,dx \qquad \displaystyle \int \frac{1}{y}\,dy \Rightarrow \log|\sin x|$

2 $\quad \displaystyle 2\left(\log|\sin x| - \log|\cos x|\right) = 2\log\left|\frac{\sin x}{\cos x}\right| = \boxed{2\log|\tan x|}$

(3) $\int \frac{\sin x}{\cos^3 x}\,dx$ $\quad y = \cos x$, $\; dy = -\sin x\,dx$ $\quad -\int \frac{1}{y^3}\,dy = -\left(-\frac{1}{2y^2}\right) = \boxed{\frac{1}{2\cos^2 x}}$

$$\int \frac{1}{\sin^3 x \cos^3 x}\,dx = \boxed{\frac{1}{2\cos^2 x} - \frac{1}{2\sin^2 x} + 2\log|\tan x|}$$