Integrali 3

Argomenti: Tecniche di integrazione **Difficoltà**: ⋆⋆⋆

Prerequisiti: Integrazione per parti

Determinare una primitiva delle seguenti funzioni. Si consiglia di fare la verifica (derivando) *prima* di andare a vedere la risposta.

Funzione	Primitiva	Funzione	Primitiva
$x \sin x$	$-x\cos x + \sin x$	xe^x	$(x-1)e^x$
$x \sinh x$	$x\cosh x - \sinh x$	$x2^{-x}$	$-\left(\frac{x}{\log 2}+\frac{1}{\log^2 2}\right)2^{-x}$
$x^2 \sin x$	$-x^2\cos x + 2x\sin x + 2\cos x$	$x^3 \cos(2x)$	$\left(\frac{1}{2}x^3-\frac{3}{4}x\right)\sin 2x+\left(\frac{3}{4}x^2-\frac{3}{8}\right)\cos 2x$
$\sin^2 x$	$\frac{x}{2}-\frac{\sin x\cos x}{2}$	$\cos^2 x$	$\frac{x}{2}+\frac{\sin x\cos x}{2}$
$\log x$	$x\log x - x$	$\arctan x$	$x\arctan x - \log\sqrt{1+x^2}$
$x \log x$	$\frac{1}{2}x^2\log x - \frac{1}{4}x^2$	$\log^2 x$	$x\log^2 x - 2x\log x + 2x$
$\log^3 x$	$x(\log^3 x - 3\log^2 x + 6\log x - 6)$	$x^7 \log x$	$\frac{x^8}{8}\log x - \frac{x^8}{64}$
$\frac{\log x}{x^2}$	$-\frac{\log x + 1}{x}$	$\frac{\log^2 x}{x^2}$	$-\frac{\log^2 x + 2\log x + 2}{x}$
$\sin^3 x$	$-\cos x + \frac{1}{3}\cos^3 x$	$\cos^3 x$	$\sin x - \frac{1}{3}\sin^3 x$
$\cos^4 x$	$\frac{3}{8}x+\frac{3}{8}\sin x\cos x+\frac{1}{4}\sin x\cos^3 x$	$\sin^5 x$	$\frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x - \cos x$
$x \log^2 x$	$x^2(2\log^2 x - 2\log x + 1)/4$	$e^x \sin x$	$e^x(\sin x - \cos x)/2$
$e^{2x}\cos(3x)$	$\frac{1}{13}e^{2x}(2\cos 3x + 3\sin 3x)$	$e^{-3x}\sin x\cos x$	$-\frac{1}{26}e^{-3x}(3\sin 2x + 2\cos 2x)$
$\log(x^2-1)$	$x\log(x^2-1) - 2x + \log\frac{x+1}{x-1}$	$\log(x^2+1)$	$x\log(x^2+1) + 2\arctan x - 2x$
$x\cos^2 x$	$\frac{x^2+2x\sin 2x+\cos^2 x}{4}$	$x\arctan x$	$\frac{1}{2}(x^2\arctan x + \arctan x - x)$
$\frac{\arctan x}{x^2}$	$-\frac{1}{x}\arctan x + \log\lvert x\rvert - \frac{1}{2}\log(x^2+1)$	$x^3\arctan x$	$\frac{1}{4}\left(x^4\arctan x - \arctan x + x - \frac{x^3}{3}\right)$
$x\arctan x^2$	$\frac{1}{2}x^2\arctan x^2 - \frac{1}{4}\log(1+x^4)$	$x\arctan^2 x$	(∗)

(∗) $\frac{1}{2}\arctan^2 x\,(x^2+1) - x\arctan x + \frac{1}{2}\log(1+x^2)$

INTEGRALI 3 Esercizi di A.M. (Parte A)

1) $x \sin x$; $\int \underset{F}{x} \underset{g}{\sin x}\, dx = -\underset{F}{x} \cdot \underset{G}{\cos x} - \int \underset{f}{1} \underset{G}{(-\cos x)}\, dx$

$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x$

Verifica $\left.\sin x - x\cos x\right]' = \cos x - (\cos x + x(-\sin x))$

$= \cancel{\cos x} - \cancel{\cos x} + x\sin x$

1b) $x \cdot e^{x}$ $\int \underset{F}{x} \underset{g}{e^{x}}\, dx = \underset{F}{x} \underset{G}{e^{x}} - \int \underset{f}{1} \underset{G}{e^{x}}\, dx = xe^{x} - e^{x}$
$= (x-1)e^{x}$

Verifica $\left[xe^{x} - e^{x}\right]' = \cancel{e^{x}} + xe^{x} - \cancel{e^{x}}$

2) $x \sinh x$

$\int \underset{F}{x} \underset{g}{\sinh x}\, dx = \underset{F}{x} \underset{G}{\cosh x} - \int \underset{f}{1} \underset{G}{\cosh x}\, dx$

$= x\cosh x - \sinh x$

Verifica $\left.x\cosh x - \sinh x\right]' = \cancel{\cosh x} + x\sinh x - \cancel{\cosh}$

2b) $x\, 2^{-x}$

$\int \underset{F}{x} \underset{g}{2^{-x}}\, dx = \underset{F}{x} \underset{G}{\left(-\frac{2^{-x}}{\log 2}\right)} - \int \underset{f}{1} \underset{G}{\left(\frac{-2^{-x}}{\log 2}\right)} dx$

$-\frac{x 2^{-x}}{\log 2} + \frac{1}{\log 2}\int 2^{-x}\, dx = -\frac{x2^{-x}}{\log 2} + \frac{-2^{-x}}{\log 2 \cdot \log 2}$

$$= -\frac{x2^{-x}}{\log 2} - \frac{2^{-x}}{\log^2 2} = -\left(\frac{x}{\log 2} + \frac{1}{\log^2 2}\right)\cdot 2^{-x}$$

3) $x^2 \sin x$

$$\int \underset{F}{x^2} \underset{g}{\sin x}\, dx = \underset{F}{x^2} \underset{G}{(-\cos x)} + \boxed{\int \underset{f}{2x} \underset{G}{(+\cos x)}\, dx}$$ ancora per parti

$$\int \underset{F}{2x} \underset{g}{\cos x}\, dx = \underset{F}{2x} \underset{G}{\sin x} - \int \underset{f}{2} \underset{G}{\sin x}\, dx$$

$$= 2x \sin x - (-2\cos x)$$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x$$

2°

3b) $\int x^3 \cos(2x)\, dx = x^3 \cdot \frac{1}{2}\sin 2x - \boxed{\int \left(3x^2 \frac{1}{2}\sin 2x\right) dx}$ ancora per parti.

2°

$$\frac{3}{2}\int x^2 \sin 2x\, dx = \frac{3}{2}\left[x^2\left(-\frac{1}{2}\cos 2x\right) - \int 2x\left(-\frac{1}{2}\cos 2x\right) dx\right.$$

3°

$$= \frac{3}{2}\left[-\frac{x^2}{2}\cos 2x + \boxed{\int x\cos 2x\, dx}\right]$$ ancora per parti

3°

$$\int x\cos 2x\, dx = x\,\frac{1}{2}\sin 2x - \int 1\cdot\frac{1}{2}\sin 2x\, dx$$

$$= \frac{1}{2}\int \sin 2x\, dx = \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) = -\frac{1}{4}\cos 2x$$

Tornando all'inizio si ha:

$$\int x^3 \cos 2x\, dx = \frac{x^3}{2}\sin 2x - \frac{3}{2}\left[-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]$$

$$= \frac{x^3}{2} \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x$$

$$= \left(\frac{1}{2} x^3 - \frac{3}{4} x \right) \sin(2x) + \left(\frac{3}{4} x^2 - \frac{3}{8} \right) \cos 2x$$

5a) $\int \log x \, dx = \int \underset{F}{\log x} \cdot \underset{g}{1} \cdot dx = \underset{F}{\log x} \cdot \underset{G}{x} - \int \underset{f}{\frac{1}{x}} \cdot \underset{G}{x} \, dx$

$= \log x \cdot x - x = x(\log x - 1)$

$= x \log x - x$

5b) $\int \arctan x \, dx = \int \underset{F}{\arctan x} \cdot \underset{g}{1} \, dx =$

$= \underset{F}{\arctan x} \cdot \underset{G}{x} - \boxed{\int \underset{f}{\frac{1}{x^2+1}} \cdot \underset{G}{x} \, dx}$

x sostituzione

$\int \frac{x}{1+x^2} dx$ pongo $y = 1 + x^2$ $\quad \frac{dy}{dx} = 2x \quad dy = 2x \, dx$

$= \frac{1}{2} \int \frac{1}{1+x^2} 2x \, dx = \frac{1}{2} \int \frac{1}{y} dy = \frac{1}{2} \log|y| = \frac{1}{2} \log|1+x^2|$

$= \frac{1}{2} \log(1+x^2)$

$\int \arctan x \, dx = x \cdot \arctan x - \frac{1}{2} \log(1+x^2)$

$= x \arctan x - \log\sqrt{1+x^2}$

6a) $\int \underset{g}{x} \, \underset{F}{\log x} = \underset{G}{\frac{1}{2} x^2} \cdot \underset{F}{\log x} - \int \frac{1}{\cancel{x}} \cdot \frac{1}{2} x^{\cancel{2}} \, dx$ $\qquad \left[\int x \, dx = \frac{1}{2} x^2 \right.$

$= \frac{1}{2} x^2 \log x - \frac{1}{2} \int x \, dx = \frac{1}{2} x^2 \log x - \frac{1}{2} \frac{1}{2} x^2$

$= \frac{1}{2} x^2 \log x - \frac{1}{4} x^2$

6b) $\int \log^2 x \, dx = \int \underset{F}{\log^2 x} \cdot \underset{g}{1} \, dx = \underset{F}{\log^2 x} \cdot \underset{G}{x} - \int \underset{f}{2 \log(x) \frac{1}{x}} \, \underset{G}{x} \, dx$

$= x \log^2 x - 2 \underbrace{\int \log x \, dx}_{\text{Vedi 5a}} = x \log^2 x - 2x \log x + 2x$

7a) $\int \log^3 x \cdot dx = \int \underset{F}{\log^3 x} \cdot \underset{g}{1} \, dx = \underset{F}{\log^3 x} \cdot \underset{G}{x} - \int 3 \log^2 x \cdot \frac{1}{x} \cdot x \, dx$

$= x \log^3 x - 3 \underbrace{\int \log^2 x \, dx}_{\text{Vedi 6b}} = x \log^3(x) - 3x \log^2 x + 6x \log x - 6x$

$= x(\log^3 x - 3 \log^2 x + 6 \log x - 6)$

7b $\int x^7 \log x \, dx = \underset{F}{\log x} \cdot \underset{G}{\frac{x^8}{8}} - \int \underset{f}{\frac{1}{\cancel{x}}} \cdot \frac{x^{\cancel{8}\,7}}{8} \, dx$

$= \frac{x^8}{8} \log x - \frac{1}{8} \int x^7 \, dx = \frac{x^8}{8} \log x - \frac{1}{8} \cdot \frac{x^8}{8}$

$= \frac{x^8}{8} \log x - \frac{x^8}{64}.$

8a) $\int \frac{\log x}{x^2}\,dx = \int \underset{F}{\log x} \cdot \underset{g}{\frac{1}{x^2}}\,dx = \underset{F}{\log x} \cdot \underset{G}{\left(-\frac{1}{x}\right)} - \int \underset{f}{\frac{1}{x}} \underset{G}{\left(-\frac{1}{x}\right)}\,dx$

$= -\frac{1}{x}\log x + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\log x - \frac{1}{x} = -\frac{\log x + 1}{x}$

8b) $\int \frac{\log^2 x}{x^2} = \int \underset{F}{\log^2 x} \underset{g}{\left(\frac{1}{x^2}\right)} = \underset{F}{\log^2 x} \underset{G}{\left(-\frac{1}{x}\right)} - \int \underset{f}{2\log x \cdot \frac{1}{x}} \underset{G}{\left(-\frac{1}{x}\right)}\,dx$

$= -\frac{1}{x}\log^2 x + 2\underbrace{\int \frac{\log x}{x^2}\,dx}_{\text{vedi 8a}} = -\frac{1}{x}\log^2 x + 2\left(-\frac{1}{x}\log x - \frac{1}{x}\right)$

$= -\frac{\log^2 x + 2\log x + 2}{x}$

9a) $\int \sin^3 x\,dx = \int \underset{F}{\sin^2 x}\, \underset{g}{\sin x}\,dx = \underset{F}{\sin^2 x} \underset{G}{(-\cos x)} - \boxed{\int \underset{f}{2\sin x\cos x} \cdot \underset{G}{(-\cos x)}}$

$+ \int 2\sin x\cos^2 x\,dx = \int 2\sin x(1-\sin^2 x)\,dx = \int (2\sin x - 2\sin^3 x)\,dx =$

$= 2\int \sin x\,dx - 2\underbrace{\int \sin^3 dx}_{\text{GRANDE RITORNO}}$

$\int \sin^3 x\,dx = -\sin^2 x\cos x + 2\int \sin x\,dx - 2\int \sin^3 x\,dx$

$3\int \sin^3 x\,dx = -\sin^2 x\cos x + 2(-\cos x) = (-1+\cos^2 x)\cos x - 2\cos x$

$= \frac{-3\cos x + \cos^3 x}{3}$; $\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x$

9b) $\int \cos^3 x\,dx = \int \underset{F}{\cos^2 x}\cdot\underset{g}{\cos x}\,dx = \underset{F}{\cos^2 x}\,\underset{G}{\sin x} - \int \underset{f}{2\cos x(-\sin x)}\cdot\underset{G}{\sin x}\,dx$

$$+\int 2\cos x\,(\overset{(1-\cos^2 x)}{\sin^2 x})\,dx = 2\int \cos x\,dx - 2\underbrace{\int \cos^3 x\,dx}_{\text{GRANDE RITORNO}}$$

$$\int \cos^3 x\,dx = \cos^2 x\,\sin x + 2\int \cos x\,dx - 2\int \cos^3 x\,dx$$

$$3\int \cos^3 x\,dx = (1-\sin^2 x)\sin x + 2\sin x$$

$$= \frac{\sin x - \sin^3 x + 2\sin x}{3} = \sin x - \frac{1}{3}\sin^3 x$$

Verifica $\left[\sin x - \frac{1}{3}\sin^3 x\right]'$

$= \cos x - \frac{1}{3}\cdot 3\sin^2 x\cos x$

$= \cos x - (1-\cos^2 x)\cos x$

$= \cancel{\cos x} - \cancel{\cos x} + \cos^3 x$ OK!

10a) $\int \cos^4 x\,dx$ 2 volte x parti con 2 Grandi Ritorni

$$\int \underset{F}{\cos^3 x}\,\underset{g}{\cos x}\,dx = \underset{F}{\cos^3 x}\,\underset{G}{\sin x} - \int 3\cos^2 x(-\sin x)\cdot\sin x\,dx$$

$$+3\int (\cos^2 x)(1-\cos^2 x)\,dx$$

$$\int \cos^4 x\,dx = \cos^3 x\sin x + 3\int \cos^2 x\,dx - 3\underbrace{\int \cos^4 x\,dx}_{1^\circ\text{ Grande Ritorno}}$$

$$4\int \cos^4 x\,dx = \cos^3 x\sin x + 3\boxed{\int \cos^2 x\,dx}$$ 2 volta per parti

$$\int \cos^2 x\,dx = \int \cos x\cos x\,dx = \cos x\sin x - \int(-\sin x)\cdot\sin x\,dx$$

$$= \cos x\sin x + \int(1-\cos^2 x)\,dx = \cos x\sin x + x - \underbrace{\int \cos^2 x\,dx}_{2^\circ\text{ Grande Ritorno}}$$

$$2\int \cos^2 x\,dx = \cos x\sin x + x = \frac{\cos x\sin x + x}{2}$$

$$4 \int \cos^4 x\,dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{2}\left(\frac{\cos x \sin x + x}{4}\right)$$

$$= \frac{3}{8}x + \frac{3}{8}\sin x \cos x + \frac{1}{4}\sin x \cos^3 x$$

lo stesso esercizio si può risolvere utilizzando le formule di duplicazione

$$\cos 2x = 2\cos^2 x - 1 \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\int \cos^4 x\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int 1 + \cos^2 2x + 2\cos x \; dx$$

$$\cos(4x) = 2\cos^2 2x - 1$$

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$= \frac{1}{4}x + \frac{1}{4}\left(2 \cdot \frac{1}{2}\sin 2x\right) + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2} + \frac{1}{2}\cos 4x \; dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\sin 4x$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$

106)

$\int \sin^5 x\,dx$ si può fare x parti + grande ritorno
2 volte per parti con precorso + grande ritorno.

oppure precorso + sostituzione.

$$\int \sin^4 x \sin x\,dx \qquad \text{pongo } y = \cos x \qquad \frac{dy}{dx} = -\sin x$$

$$\int \sin x \cdot \left(1-\cos^2 x\right)^2 dx \qquad dy = -\sin x\,dx$$

dy con aggiunta di un segno −

$$= -\int (1-y^2)^2 \, dy = -\int (1+y^4-2y^2) \, dy$$

$$= -y - \frac{1}{5}y^5 + \frac{2}{3}y^3 \quad \text{torno in } \cos x$$

$$= -\cos x - \frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x$$

11a) $$\int x \log^2 x \, dx = \int \underbrace{\log^2 x}_{F} \; \underbrace{x}_{g} = \underbrace{\log^2 x}_{F} \cdot \underbrace{\frac{x^2}{2}}_{G} - \int \underbrace{\cancel{2}\log x \cdot \frac{1}{\cancel{x}}}_{f} \cdot \underbrace{\frac{x^{\cancel{2}}}{\cancel{2}}}_{G}$$

$$= \frac{x^2}{2}\log^2 x - \int \underbrace{x \log x}_{\text{Vedi esercizio 6a}} \, dx = \frac{x^2}{2}\log^2 x - \frac{x^2}{2}\log x + \frac{1}{4}x^2$$

$$= \frac{x^2}{4}\left(2\log^2 x - 2\log x + 1\right)$$

11b) $$\int \underbrace{e^x}_{F} \underbrace{\sin x}_{g} \, dx = \underbrace{e^x}_{F} \underbrace{(-\cos x)}_{G} - \int e^x(-\cos x) \, dx$$

$$= -e^x \cos x + \boxed{\int e^x \cos x \, dx} \quad \text{parti}$$

$$\left[\; e^x \sin x - \int e^x \sin x \, dx \; ;\right.$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{\text{grande ritorno}}$$

$$\int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2}$$

12a $\int \underset{F}{e^{2x}} \cdot \underset{g}{\cos(3x)}\,dx = \underset{F}{e^{2x}} \cdot \underset{G}{\frac{1}{3}\sin 3x} - \int \underset{f}{2e^{2x}} \cdot \underset{G}{\frac{1}{3}\sin 3x}\,dx$

$= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\boxed{\int e^{2x}\sin 3x\,dx}$ per parti

$\int \underset{F}{e^{2x}}\,\underset{g}{\sin 3x}\,dx = e^{2x}\cdot\left(-\frac{1}{3}\cos 3x\right) + \int 2e^{2x}\cdot\frac{1}{3}\cos 3x\,dx$

$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[-\frac{1}{3}e^{2x}\cos 3x + \underbrace{\frac{2}{3}\int e^{2x}\cos 3x\,dx}_{\text{Grande ritorno}}\right]$

$\frac{13}{9}\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$

$= \frac{9}{13}e^{2x}\left(\frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x\right) = \frac{1}{13}e^{2x}\left(3\sin 3x + 2\cos 3x\right)$

12b) $\int e^{-3x}\sin x\cos x$ $\quad 2\sin x\cos x = \sin 2x \quad \sin x\cos x = \frac{\sin 2x}{2}$

$= \frac{1}{2}\int \underset{g}{e^{-3x}}\,\underset{F}{\sin 2x}\,dx = \underset{F}{\sin 2x}\left(-\frac{1}{3}e^{-3x}\right) - \int \underset{f}{2\cos 2x}\,\underset{G}{\left(-\frac{1}{3}e^{-3x}\right)}\,dx$

lo moltiplico alla fine

$= -\frac{1}{3}e^{-3x}\sin 2x + \frac{2}{3}\boxed{\int e^{-3x}\cos 2x\,dx}$

$\int e^{-3x}\cos 2x\,dx = \cos 2x\left(-\frac{1}{3}e^{-3x}\right) - \int -2\sin 2x\left(-\frac{1}{3}e^{-3x}\right)dx$

$= -\frac{1}{3}e^{-3x}\cos 2x - \frac{2}{3}\underbrace{\int e^{-3x}\sin 2x\,dx}$ Grande Ritorno

$\int e^{-3x}\sin 2x = -\frac{1}{3}e^{-3x}\sin 2x + \frac{2}{3}\left[-\frac{1}{3}e^{-3x}\cos 2x - \frac{2}{3}\int e^{-3x}\sin 2x\,dx\right]$

$\frac{13}{9}\int e^{-3x}\sin 2x\,dx = -\frac{1}{26} = \boxed{\frac{9}{13}\cdot\frac{1}{2}}\,e^{-3x}\left(3\,\cancel{\tfrac{1}{3}}\sin 2x - \frac{2}{\cancel{9}}\cos 2x\right)$

iniziale

13a) $\int \log(x^2-1)\,dx = \int \underset{F}{\log(x^2-1)} \cdot \underset{g}{1} =$

$= \underset{G}{x} \cdot \underset{F}{\log(x^2-1)} - \boxed{\int \underbrace{\frac{2x}{x^2-1}}_{f} \cdot \underset{G}{x}\,dx}$

$2\int \frac{x^2}{x^2-1} = 2\int \frac{x^2+1-1}{x^2-1} = \boxed{2\int \left(\frac{x^2-1}{x^2-1} + \frac{1}{(x-1)(x+1)}\right) dx}$

$\frac{1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$

$Ax + A + Bx - B$

$A+B=0 \quad 1+B+B=0$

$A-B=1 \quad A=1+B$

$2B=-1$

$B=-1/2$

$A=1/2$

$= \frac{1/2}{x-1} - \frac{1/2}{x+1} = -2\int dx - \int \frac{1}{x-1}dx + \int \frac{1}{x+1}dx$

$\int \log(x^2-1)\,dx = \boxed{x\log(x^2-1) - 2x + \log\frac{x+1}{x-1}}$ Ris.

13b) $\int \log(x^2+1)\,dx = \int \underset{F}{\log(x^2+1)} \cdot \underset{g}{1}\,dx = \underset{G}{x}\,\underset{F}{\log(x^2+1)} \boxed{-2\int \frac{x}{x^2+1}\cdot x\,dx}$

$-2\int \frac{x^2+1-1}{x^2+1}dx = -2\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\,dx = -2\int 1 - \frac{1}{x^2+1}\,dx$

$= \boxed{-2x + \arctan x}$

$=$

$\int \log(x^2+1)\,dx = \boxed{x\log(x^2+1) - 2x + 2\arctan(x)}$

14a) $\int x\cos^2 x \, dx \qquad \cos 2x = 2\cos^2 x - 1 \qquad \cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$

$$\int \frac{x}{2} + \frac{x\cos 2x}{2} \, dx = \frac{1}{2}\int \left(x + \boxed{x\cos 2x}\right) dx = \frac{1}{2}\,\frac{x^2}{2} + \frac{1}{2}\int x\cos 2x \, dx$$

x part.

$$\int \underset{F}{x}\,\underset{g}{\cos 2x} \, dx = \underset{F}{x}\,\underset{G}{\frac{1}{2}\sin 2x} - \int \underset{f}{1}\cdot\underset{G}{\frac{1}{2}\sin 2x} \, dx$$

$$= \frac{x}{2}\sin 2x - \left(-\frac{1}{4}\cos 2x\right)$$

$$\int x\cos^2 x \, dx = \frac{1}{2}\left(\frac{x^2}{2} + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right) = \boxed{\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x}$$

14b)

$$\int \underset{g}{x}\,\underset{F}{\arctan x} \, dx = \underset{F}{\arctan x}\cdot\underset{G}{\frac{x^2}{2}}\boxed{- \int \frac{1}{1+x^2}\cdot\frac{x^2}{2} \, dx}$$

$$-\frac{1}{2}\int \frac{x^2+1-1}{1+x^2} = -\frac{1}{2}\int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \, dx$$

$$= -\frac{1}{2}\int 1 \, dx - \frac{1}{2}\int -\frac{1}{1+x^2} \, dx$$

$$\int x\arctan x = \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x$$

$$= \boxed{\frac{1}{2}\left(x^2\arctan x + \arctan x - x\right)}$$

15a) $$\int \frac{\arctan x}{x^2}\,dx = \int \underset{F}{\arctan x} \cdot \underset{g}{\frac{1}{x^2}}\,dx = \underset{G}{-\frac{1}{x}}\,\underset{F}{\arctan x} \boxed{- \int \frac{1}{1+x^2} \cdot \left(-\frac{1}{x}\right) dx}$$

$$\int \frac{1+x^2-x^2}{x(x^2+1)}\,dx = \int \frac{\cancel{1+x^2}}{x(\cancel{1+x^2})} - \frac{x^{\cancel{2}}}{\cancel{x}(1+x^2)}$$

$$= \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1} = \log|x| - \frac{1}{2}\log(x^2+1)$$

$$\int \frac{\arctan x}{x^2} = \boxed{-\frac{1}{x}\arctan x - \log|x| - \frac{1}{2}\log(1+x^2)}$$

15b) $$\int \underset{g}{x^3}\,\underset{F}{\arctan x} = \underset{G}{\frac{x^4}{4}}\,\underset{F}{\arctan x} - \int \frac{1}{1+x^2} \cdot \frac{x^4}{4}\,dx$$

$$= \frac{x^4}{4}\arctan x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx \qquad \frac{P(x)}{Q(x)}$$

Divisione $P(x)/Q(x)$

$$\begin{array}{r|l} x^4 & x^2+1 \\ \hline -x^4 - x^2 & x^2 - 1 = A(x) \\ \hline // \quad -x^2 & \\ +x^2+1 & \\ \hline // \quad 1 = R(x) & \end{array}$$

$$\frac{P(x)}{Q(x)} = \frac{Q(x)\cdot A(x) + R(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$$

$$\frac{x^4}{1+x^2} = x^2 - 1 + \frac{1}{1+x^2}$$ da cui $$-\frac{1}{4}\int x^2\,dx - \int 1\,dx + \int \frac{1}{1+x^2}$$

$$= -\frac{1}{4}\left(\frac{x^3}{3} - x + \arctan x\right) \qquad \boxed{\frac{1}{4}\left(x^4\arctan x - \arctan x + x - \frac{x^3}{3}\right)}$$

$$\int x^3 \arctan x\,dx = \frac{x^4}{4}\arctan x - \frac{1}{12}x^3 + \frac{x}{4} - \frac{1}{4}\arctan x$$

16a) $\int \underset{g}{x} \underset{F}{\arctan x^2}$

Calcolo derivata di $\arctan x^2$

la $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$D[f(g(x)) = g'(x)\cdot f'(g(x))]$ si ha $f'(g(x)) = \frac{1}{1+(x^2)^2} = \frac{1}{1+x^4}$

si ha $g(x) = x^2 \quad \frac{d}{dx}(x^2) = 2x = g'(x)$

$$\frac{d}{dx}\arctan(x^2) = \frac{2x}{1+x^4}$$

$$\int x\arctan x^2 = \arctan x^2\cdot\frac{x^2}{2} - \int\frac{\cancel{2}x}{1+x^4}\cdot\frac{x^2}{\cancel{2}}$$

$$= \frac{x^2}{2}\arctan x^2 - \frac{1}{4}\int\frac{4x^3}{1+x^4} = \boxed{\frac{1}{2}x^2\arctan x^2 - \frac{1}{4}\log(1+x^4)}$$

16b) $\int x\arctan^2 x = \frac{x^2}{2}\arctan^2 x - \int 2\arctan x\cdot\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$

$$= \frac{x^2}{2}\arctan^2 x - \boxed{\int \underset{F}{\arctan x}\cdot\underset{g}{\frac{x^2}{1+x^2}}\,dx}$$

sapendo che:

$$\frac{x^2}{1+x^2} \to \int\frac{x^2+1-1}{1+x^2}\,dx = \int 1\,dx - \int\frac{1}{1+x^2}\,dx = x - \arctan x$$

$$= \frac{x^2}{2}\arctan^2 x - \left[\underset{F}{\arctan x}\cdot\underset{G}{(x-\arctan x)} - \int\underset{f}{\frac{1}{1+x^2}}\cdot\underset{G}{(x-\arctan x)}\,dx\right.$$

$$= \frac{x^2}{2}\arctan^2 x - x\arctan x + \arctan^2 x + \frac{1}{2}\int\frac{2x}{1+x^2} - \int\arctan x\cdot\frac{1}{1+x^2}\,dx$$

Sapendo che per sostituzione

$$\int \frac{\arctan x}{1+x^2}\,dx \qquad y = \arctan x \qquad \frac{dy}{dx} = \frac{1}{1+x^2}; \quad dy = \frac{1}{1+x^2}\,dx$$

$$\int y\,dy = y^2/2 \rightarrow \text{torno in } x \quad \int \frac{\arctan x}{1+x^2}\,dx = \frac{\arctan^2(x)}{2}$$

$$= \frac{x^2}{2}\arctan^2 x - x\arctan x + \arctan^2 x + \frac{1}{2}\log(1+x^2) - \frac{\arctan^2 x}{2}$$

$$= \boxed{\frac{1}{2}(x^2+1)\arctan^2 x - x\arctan x + \frac{1}{2}\log(1+x^2)}$$

4a) $$\int \sin^2 x\,dx \qquad \cos 2x = 1 - 2\sin^2 x \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$\sin 2x = 2\sin x\cos x; \qquad \int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{2}\sin x\cos x = \boxed{\frac{x}{2} - \frac{\sin x\cos x}{2}}$$

4b) $$\int \cos^2 x\,dx; \qquad \cos 2x = 2\cos^2 x - 1 \qquad \cos 2x = \frac{1+\cos 2x}{2}$$

$$\frac{1}{2}\int 1\,dx + \frac{1}{2}\int \cos 2x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x = \boxed{\frac{x}{2} + \frac{\sin x\cos x}{2}}$$