1. Let us consider the functionals

$$F(u) = u(0) + \int_0^1 (\dot{u}^2 + u)\, dx, \qquad G(u) = [u(0)]^3 + \int_0^1 (\dot{u}^2 + u)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = 3$.

(b) Discuss the minimum problem for $G(u)$ with boundary condition $u(1) = 3$.

(a) $F(u + tv) = u(0) + tv(0) + \int_0^1 (\dot{u}^2 + u) + t\int_0^1 (2\dot{u}\dot{v} + v) + t^2 \int_0^1 \dot{v}^2$

Therefore, any min. point $u_0(x)$ satisfies

$$\delta F(u_0, v) = v(0) + \int_0^1 (2\dot{u}_0 \dot{v} + v) = 0$$

Integrating by parts we obtain that

$$v(0) + 2\dot{u}_0(1)\underbrace{v(1)}_{0} - 2\dot{u}_0(0)v(0) + \int_0^1 (-2\ddot{u}_0 + 1)v = 0$$

From this relation we deduce in the usual way that u_0 satisfies

$$\begin{cases} \ddot{u}_0 = \frac{1}{2} \\ u_0(1) = 3 \\ \dot{u}_0(0) = \frac{1}{2} \end{cases}$$

$u_0(x) = \frac{x^2}{4} + ax + b$ $\qquad$ $\dot{u}_0(x) = \frac{x}{2} + a$

$a = \frac{1}{2}$ $\qquad$ $b = 3 - \frac{1}{2} - \frac{1}{4} = \frac{9}{4}$ $\rightsquigarrow$ $u_0(x) = \frac{x^2 + 2x + 9}{4}$

The system has a unique solution $u_0(x)$. This is the unique min. point, as we can prove by computing $F(u_0 + v)$ where $v(x)$ is any admissible variation, namely $v \in C^1([0,1])$ with $v(1) = 0$.

↑ or even $H^1((0,1))$

(b) The minimum does NOT exist, and the infimum is $-\infty$.

A possible minimizing sequence is $u_n(x) = n(x-1) + 3$, because

$$G(u_n) = (3-n)^3 + O(n^2) \quad \text{as} \quad n \to +\infty.$$

— o — o —

Alternative for point (a): observe that $\int_0^1 \dot{u} = u(1) - u(0) = 3 - u(0)$, so that

$$F(u) = 3 + \int_0^1 (\dot{u}^2 - \dot{u} + u).$$ Then proceed with this new functional.

— o — o —

2. Discuss existence, uniqueness and regularity of solutions to the boundary value problem

$$u'' = -1 + \sqrt{u}, \qquad u(0) = 1/2, \quad \dot{u}(2020) = 1.$$

See IstAM_20_CS3, problem 2.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell [\sin(\dot u^2) - \cos(u) - \arctan(u^4)]\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine for which values of ℓ the infimum is actually a minimum.

(a) $\sin(p^2) - \cos(s) - \arctan(s^4) \sim p^2 - 1 + \frac{s^2}{2} + o(p^2+s^2)$ as $(s,p) \to (0,0)$.

As a consequence,

$$\delta F(u_0, v) = \int_0^\ell (2\dot v^2 + v^2) \geq 0 \qquad \forall \text{ admissible } v.$$

Since $u_0(x)$ satisfies (E) + (L+) + (J+), it follows that $u_0(x)$ is a WLM $\boxed{\text{for every } \ell > 0.}$

(b) $u_0(x)$ is $\boxed{\text{NEVER}}$ a SLM. To this end, it is enough to observe that

$$E(x,s,p,q) = L(x,s,p+q) - L(x,s,p) - q L_p(x,s,p)$$
$$= \sin[(p+q)^2] - \sin(p^2) - 2p\cos(p^2)\cdot q$$

When $s = p = 0$, this becomes $\sin(q^2)$, which is not ≥ 0 for every $q \in \mathbb{R}$.

Since (W) is violated, $u_0(x)$ is not a SLM.

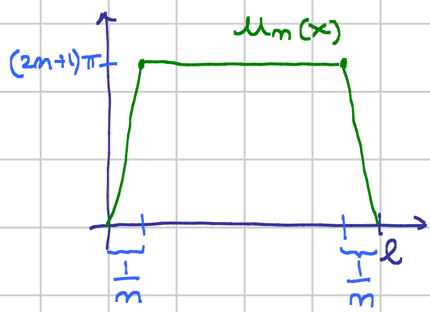
(c) We claim that the infimum is $-\left(\frac{\pi}{2}+2\right)\ell$ for every $\ell > 0$, and it is $\boxed{\text{NEVER}}$ a minimum

Indeed the infimum coincides with the infimum of the relaxed functional, which is

$$-\int_0^\ell (+1 + \cos(u) + \arctan(u^4))$$

with $\boxed{\text{LOSS of the BCs}}$. For this functional a possible min. sequence is $u_m(x) = (2m+1)\pi$.

If we do not want to neglect BCs, we can define $u_m(x)$ as in the figure



— o — o —

4. For every real number $\ell > 0$, let us set

$$m(\ell) := \min\left\{\int_0^\ell (\dot{u}^4 - u^2)\,dx : u \in C^1([0,\ell]),\ \int_0^\ell u(x)\,dx = 2020\right\}.$$

(a) Prove that $m(\ell)$ is well-defined and negative for every $\ell > 0$.

(b) Prove that $m(\ell) \to -\infty$ as $\ell \to +\infty$.

(c) Determine all real numbers α such that

$$\lim_{\ell\to+\infty} \frac{m(\ell)}{\ell^\alpha} \in (-\infty, 0).$$

(a) The infimum is negative because the constant function $u_0(x) = \frac{2020}{\ell}$ is a competitor.

Existence of the minimum follows from the direct method. In order to obtain compactness, we need to combine three ingredients:

→ the usual Hölder estimate $|u(x)| \le |u(x_0)| + \|\dot{u}\|_{L^2}\, |x - x_0|^{1/2}$

→ the existence of $x_0 \in (0, \ell)$ where $u(x_0) = 2020/\ell$

→ $L(p,s) = p^4 - s^2$ and $4 > 2$.

(b) A possible competitor is $u_0(x) = \frac{2020}{\ell} + \left(x - \frac{\ell}{2}\right)$. In this case

$$\int_0^\ell \dot{u}_0^4 \sim \text{cost}\cdot\ell \qquad \int_0^\ell u_0^2 \sim \text{cost}\cdot\ell^3.$$

(c) Let us set $v(x) = u(\ell x)$, so that $u(x) = v\left(\frac{x}{\ell}\right)$ and $\dot{u}(x) = \frac{1}{\ell}\dot{v}\left(\frac{x}{\ell}\right)$

With a variable change we obtain that

$$\int_0^\ell (\dot{u}(x)^4 - u(x)^2) \underset{x = \ell y}{=} \int_0^1 (\dot{u}(\ell y)^4 - u(\ell y)^2)\,\ell\,dy = \int_0^1 \left\{\frac{1}{\ell^4}\dot{v}(y)^4 - v(y)^2\right\}\ell\,dy$$

and similarly

$$\int_0^\ell u(x)\,dx = \int_0^1 u(\ell y)\,\ell\,dy \rightsquigarrow \int_0^1 v(y)\,dy = \frac{2020}{\ell}$$

Now we set $v(y) = \ell^2 w(y)$ and we obtain

$$\int_0^\ell (\dot{u}(x)^4 - u(x)^2) = \ell^5 \int_0^1 \{\dot{w}(y)^4 - w(y)^2\} \quad \text{and} \quad \int_0^1 w(y)\,dy = \frac{2020}{\ell^3} \to 0$$

It follows that

$$\lim_{\ell\to+\infty} \frac{m(\ell)}{\ell^\alpha} = \begin{cases} -\infty & \text{if } \alpha < 5 \\ 0 & \text{if } \alpha > 5 \\ \underbrace{\min\left\{\int_0^1 (\dot{w}^4 - w^2) : \int_0^1 w = 0\right\}}_{\in(-\infty,0)\ :\ \text{Direct method} + u_\varepsilon(x) = \varepsilon\left(x - \frac{1}{2}\right)} & \boxed{\text{if } \alpha = 5} \end{cases}$$

— o — o —