Università di Pisa – Corso di Laurea in Matematica

Exam paper of “Elementi di Calcolo delle Variazioni”

Pisa, 21 February 2020

1. Let us consider the functionals

$$F(u) = u(0) + \int_0^1 \left(\dot{u}^2 + u\right)\,dx, \qquad G(u) = [u(0)]^3 + \int_0^1 \left(\dot{u}^2 + u\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = 3$.

   (b) Discuss the minimum problem for $G(u)$ with boundary condition $u(1) = 3$.

2. Discuss existence, uniqueness and regularity of solutions to the boundary value problem

$$u'' = -1 + \sqrt{u}, \qquad u(0) = 1/2, \quad \dot{u}(2020) = 1.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left[\sin(\dot{u}^2) - \cos(u) - \arctan(u^4)\right]\,dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Determine for which values of ℓ the infimum is actually a minimum.

4. For every real number $\ell > 0$, let us set

$$m(\ell) := \min\left\{\int_0^\ell \left(\dot{u}^4 - u^2\right)\,dx : u \in C^1([0,\ell]),\ \int_0^\ell u(x)\,dx = 2020\right\}.$$

   (a) Prove that $m(\ell)$ is well-defined and negative for every $\ell > 0$.

   (b) Prove that $m(\ell) \to -\infty$ as $\ell \to +\infty$.

   (c) Determine all real numbers α such that

$$\lim_{\ell \to +\infty} \frac{m(\ell)}{\ell^\alpha} \in (-\infty, 0).$$

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.