1. Determine for which values of the real parameter a the problem

$$\min\left\{\int_{-\pi}^{\pi}\{(\dot u-\cos x)^2+(u-\sin x)^2\}\,dx : u\in C^1([-\pi,\pi]),\ u(0)=a\right\}$$

admits a solution (note that the condition is given in the midpoint of the interval).

The problem admits a solution in $C^1([-\pi,\pi])$ $\boxed{\text{if and only if } a=0.}$

If $a=0$ the solution is $u(x)=\sin x$ (trivial).

If $a\neq 0$, we consider separate min pbms in the intervals $[-\pi,0]$ and $[0,\pi]$.

We obtain in the usual way ELE of the form

$\ddot u = u - 2\sin x$ $\qquad\qquad (\dot u-\cos x)' = u-\sin x$

with BCs, respectively

$u(0)=a$ $\qquad\qquad u(0)=a$

$\dot u(-\pi)=-1$ $\qquad\qquad \dot u(\pi)=1$ $\qquad\qquad L_p'=0$ for $x=-\pi$ or $x=\pi$

The general solution of the equation is

$$\boxed{u(x)=\sin x+\lambda\cosh x+\mu\sinh x}$$

so that BCs become $\boxed{\lambda=a}$ (in both cases) and, since

$$\dot u(x)=\cos x+\lambda\sinh x+\mu\cosh x,$$

$\lambda\sinh(-\pi)+\mu\cosh(-\pi)=0 \quad\rightsquigarrow\quad \boxed{\mu=a\tanh(\pi)}$

$\lambda\sinh(\pi)+\mu\cosh(\pi)=0 \quad\rightsquigarrow\quad \boxed{\mu=-a\tanh(\pi)}$

Since $\dot u(0)=1+\mu$, the two separate solutions do NOT glue in a C^1 way in $x=0$.

Now one can conclude in at least two different ways:

1 - by observing that the glueing is the <u>unique</u> minimum point in a larger ambient space, for example H^1 or piecewise C^1 (this requires some convexity argument);

2 - by observing that a C^1 minimum point to the original pbm must coincide, on the two subintervals, with the two functions that we found above.

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2. Discuss existence, uniqueness, and regularity of functions $u : \mathbb{R} \to \mathbb{R}$ that are *periodic* and satisfy

$$u'' = u^3 + \sin^2 x \qquad \forall x \in \mathbb{R}.$$

The function $\sin^2 x$ is periodic with minimal period π. Let us consider the pbm

$$\min\left\{ \int_0^{\pi} \left(\frac{1}{2}\dot{u}^2 + \frac{1}{4}u^4 + \sin^2 x\, u \right) dx \;:\; u \in H^1((0,\pi)),\ u(0) = u(\pi) \right\}$$

A standard application of the direct method proves that the pbm has a unique minimizer $u_0(x)$ of class C^∞. Indeed

- there exists $A \in \mathbb{R}$ such that
$$\frac{1}{2}p^2 + \frac{1}{4}s^4 + \sin^2 x \cdot s \geq \frac{1}{2}p^2 + \frac{1}{8}s^4 - A \qquad \forall (x,s,p)$$
and this implies compactness in a usual way
- the strict convexity of the Lagrangian in (s,p) implies both LSC and uniqueness
- regularity follows in the usual way (initial step + bootstrap).

Moreover, the unique minimizer $u_0(x)$ satisfies the diff. equ. in $(0,\pi)$ and

$u_0(0) = u_0(\pi)$ and $\dot{u}_0(0) = \dot{u}_0(\pi)$ (BC generated "on the road")

We claim that $u_0(x)$, extended to $\mathbb{R}$ by periodicity, is the UNIQUE periodic solution to the diff. equ.

→ It is a solution in every interval $(k\pi, (k+1)\pi)$ because in this interval

$$\ddot{u}_0(x) \underset{\text{periodicity}}{=} \ddot{u}_0(x-k\pi) \underset{\text{equ. in } (0,\pi)}{=} u_0(x-k\pi)^3 + \sin^2(x-k\pi) \underset{\text{periodicity of } u_0 \text{ and } \sin^2}{=} u_0(x)^3 + \sin^2 x$$

Moreover $u_0(k\pi) = u_0(0) = u_0(\pi) = u_0((k+1)\pi)$ and the same for $\dot{u}_0$.

→ It is unique because of two facts.

- If $u(x)$ is a T-periodic solution, then $\sin^2 x$ is T-periodic, and therefore $T = k\pi$ for some positive integer k
- Every $k\pi$-periodic solution is a minimizer for the functional in $(0,k\pi)$ with PBC (due to convexity of the Lagrangian), but we know that the minimizer is unique (again strict convexity) and we already have one solution / minimizer, namely $u_0(x)$ extended by periodicity.

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left[\sin(\dot u^2)+\cos(u)-\arctan(u^4)\right]\,dx : u\in C^1([0,\ell]),\ u(0)=u(\ell)=0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

(a) $\boxed{u_0(x) \text{ is WLM} \iff \ell < \pi\sqrt{2}}$ Indeed $\delta^2 F(u_0,v) = \int_0^\ell 2\dot v^2 - v^2$

and JDE is $\ddot u = -\frac{1}{2}u$ with solution $\sin\left(\frac{1}{\sqrt{2}}x\right)$.

From the classical theory it follows that

- if $\ell < \pi\sqrt{2}$, then u_0 satisfies $(E)+(L^+)+(J^+) \Rightarrow u_0$ is WLM
- if $\ell > \pi\sqrt{2}$, then u_0 does not satisfy $(J) \Rightarrow u_0$ is not WLM
- if $\ell = \pi\sqrt{2}$, then

$$F(u) < \int_0^\ell \dot u^2 + 1 - \frac{1}{2}u^2 \le F(u_0) \qquad \forall u \neq u_0$$

$\sin x^2 \le x^2$, $\cos x > 1-\frac{x^2}{2}$ if $x \neq 0$

and therefore again u_0 is not WLM

(b) $\boxed{u_0(x) \text{ is not SLM for every } \ell > 0}$ Indeed (W) is not satisfied because

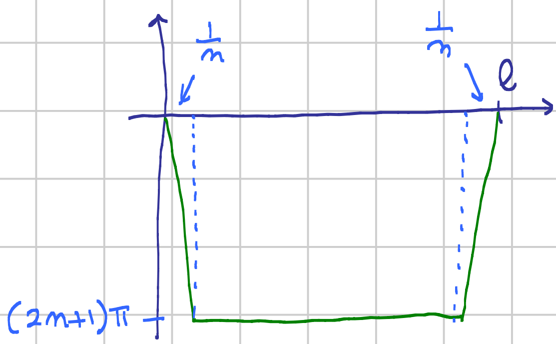
$$E(x,s,p,q) = \sin((p+q)^2) - \sin(p^2) - 2qp\cos(p^2) < 0 \quad \text{if } p=0 \text{ and } q=\sqrt{\frac{3}{2}}.$$

(c) $\boxed{\inf = -\left(2+\frac{\pi}{2}\right)\ell \quad \text{for every } \ell > 0}$ The inequality $\ge$ is trivial.

For the reverse inequality, we observe that the infimum coincides with the infimum of the relaxed functional, namely

$$\bar F(u) = \int_0^\ell \left(-1+\cos u - \arctan u^4\right)dx$$

for which we have minimizing sequences as in the figure



4. For every real number $\ell > 0$, and every real number α, let us set

$$I(\alpha, \ell) := \inf \left\{ \int_0^\ell \left(\dot{u}^2 - u^2\right) dx : u \in C^1([0,\ell]),\ \int_0^\ell u(x)\,dx = \alpha \right\}.$$

(a) Determine whether there exists $\ell > 0$ such that $I(0, \ell) = 0$.

(b) Determine whether there exists $\ell > 0$ such that $I(0, \ell) = -\infty$.

(c) Determine whether there exists $\ell > 0$ such that $I(2020, \ell) = -\infty$.

(a) $\boxed{\text{YES}}$ More precisely, $I(0,\ell) = 0$ when ℓ is small enough. Indeed from the zero average we deduce that $u(c) = 0$ for some $c \in (0,1)$, and therefore $|u(x)| \le |x-c|^{1/2} \|\dot{u}\|_{L^2} \le \sqrt{\ell}\, \|\dot{u}\|_{L^2}$, which implies that

$$\int_0^\ell u(x)^2\,dx \underset{\substack{\uparrow \\ \text{previous estimate}}}{\le} \int_0^\ell \ell \left(\int_0^\ell \dot{u}(x)^2\,dx \right) dx \underset{\substack{\uparrow \\ \text{computation}}}{\le} \ell^2 \int_0^\ell \dot{u}(x)^2\,dx \underset{\substack{\uparrow \\ \text{if } \ell \le 1}}{\le} \int_0^\ell \dot{u}(x)^2\,dx$$

(b) $\boxed{\text{YES}}$ More precisely, $I(0,\ell) = +\infty$ when ℓ is large enough.
Indeed, let us consider the function $u(x) = x - \frac{\ell}{2}$. Easy computations show that $\int_0^\ell u = 0$, $\int_0^\ell \dot{u}^2 = \ell$, $\int_0^\ell u^2$ grows quadratically in ℓ.

(c) $\boxed{\text{YES}}$ As before, consider the function $u(x) = \frac{2020}{\ell} + (x - \ell)$.

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