Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 11 January 2020

1. Determine for which values of the real parameter a the problem

$$\min\left\{\int_{-\pi}^{\pi}\left\{(\dot{u}-\cos x)^2+(u-\sin x)^2\right\}\,dx : u\in C^1([-\pi,\pi]),\ u(0)=a\right\}$$

admits a solution (note that the condition is given in the midpoint of the interval).

2. Discuss existence, uniqueness, and regularity of functions $u:\mathbb{R}\to\mathbb{R}$ that are *periodic* and satisfy

$$u''=u^3+\sin^2 x \qquad \forall x\in\mathbb{R}.$$

3. Let us consider, for every $\ell>0$, the problem

$$\inf\left\{\int_0^{\ell}\left[\sin(\dot{u}^2)+\cos(u)-\arctan(u^4)\right]\,dx : u\in C^1([0,\ell]),\ u(0)=u(\ell)=0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a strong local minimum.

   (c) Determine the infimum as a function of ℓ.

4. For every real number $\ell>0$, and every real number α, let us set

$$I(\alpha,\ell):=\inf\left\{\int_0^{\ell}\left(\dot{u}^2-u^2\right)\,dx : u\in C^1([0,\ell]),\ \int_0^{\ell}u(x)\,dx=\alpha\right\}.$$

   (a) Determine whether there exists $\ell>0$ such that $I(0,\ell)=0$.

   (b) Determine whether there exists $\ell>0$ such that $I(0,\ell)=-\infty$.

   (c) Determine whether there exists $\ell>0$ such that $I(2020,\ell)=-\infty$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.