ESERCIZI SU INTEGRALI DOPPI

• $D = \{ [0,1] \times [0,1] \}$

$f(x,y) = x+y$

$$\int_0^1 \int_0^1 x+y = \int_0^1 dx \int_0^1 x+y \, dy = \int_0^1 xy + \frac{y^2}{2} \Big|_0^1 dx = \int_0^1 \left(x + \frac{1}{2} \right) dx$$

$$= \frac{x^2}{2} + \frac{1}{2} x \Big|_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

• $D = \{ [1,2] \times [0,3] \}$

$$\int_1^2 \int_0^3 x+y \; dx\,dy = \int_1^2 dx \int_0^3 (x+y)\, dy = \int_1^2 dx \left[xy + \frac{y^2}{2} \right|_0^3 =$$

$$= \int_1^2 \left[3x + \frac{9}{2} \right] dx = \frac{3}{2}x^2 + \frac{9}{2} x \Big|_1^2 = 6 + 9 - \frac{3}{2} - \frac{9}{2} =$$

$$= \frac{12 + 18 - 3 - 9}{2} = 9$$

• $f(x,y) = x^2 y$

▶ $D = [0,1] \times [0,3]$

$$\int_0^1 dx \int_0^3 x^2 y \, dy = \int_0^1 dx \left[x^2 \frac{y^2}{2} \right|_0^3 = \int_0^1 \frac{9}{2} x^2 \, dx = \frac{3}{2} x^3 \Big|_0^1 = \frac{3}{2}$$

sicuro sia questa la primitiva?

▶ $D = [1,2] \times [2,3]$

$$\int_1^2 dx \int_2^3 dy \left[x^2 y \right] = \int_1^2 dx \left[x^2 \frac{y^2}{2} \right|_2^3 = \int_1^2 \left[\frac{9}{2} x^2 - \frac{4}{2} x^2 \right] dx =$$

$$= \frac{5}{6} x^3 \Big|_1^2 = \frac{40}{6} - \frac{5}{6} = \frac{35}{6}$$

NOTA

$$\int \overset{f}{\log y} \cdot \overset{1}{dy} = y \log y - \int \frac{1}{y} y \, dy = y(\log y - 1)$$

• $f(x,y) = \log y$

$$\int_0^1 dx \int_1^2 \log y \, dy =$$

$$= \int_0^1 dx \left[y \log y - 1 \right|_1^2 = \int_0^1 2(\log 2 - 1) - 1(\cancel{\log 1} - 1) \, dx$$

Occhio questa non è la primitiva che hai scritto prima

$$= 2\log 2 - 2 + 1 = 2 \log 2 - 1$$

- $f(x,y) = \frac{x}{y}$

Sicuro di aver messo gli estremi corretti? N.B. se fossero questi sarebbe un integrale improprio divergente

$D = \{ [0,1] \times [0,2] \}$

$$\int_0^1 \int_0^2 \frac{x}{y}\, dx\, dy = \int_0^1 dx \int_0^2 x \cdot \frac{1}{y}\, dy = \int_0^1 x \log y \Big|_0^2 dx = \frac{1}{2} \log 2 = \log \sqrt{2}$$

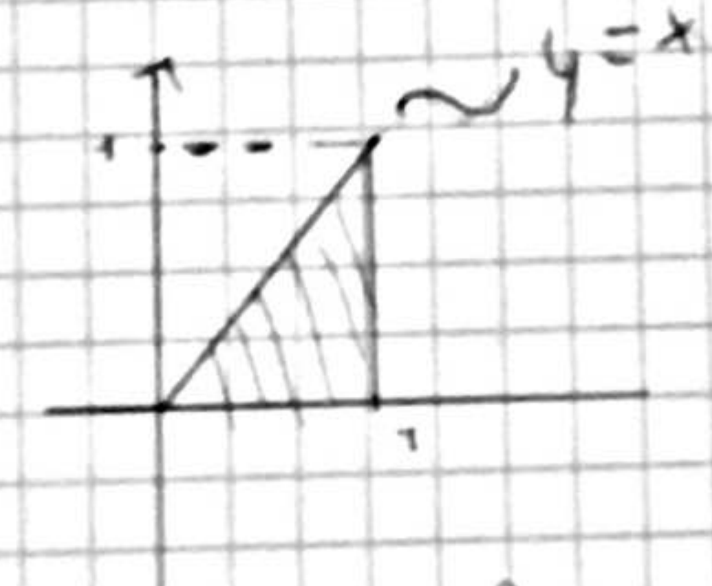
Log di -1? la primitiva di 1/x è log(|x|)…

$D = \{ [-2,-1] \times [-2,-1] \}$

$$\int_{-2}^{-1} dx \int_{-2}^{-1} x \cdot \frac{1}{y}\, dy = \int_{-2}^{-1} x \log y \Big|_{-2}^{-1} dx = \frac{x^2}{2} \Big[\log(-1) - \log(-2) \Big] \Big|_{-2}^{-1}$$

$$= \frac{x^2}{2} \log(1/2) \Big|_{-2}^{-1} = \frac{1}{2} - \frac{4}{2} \left[\log\left(\frac{1}{2}\right) \right] = \frac{3}{2} \log 2$$

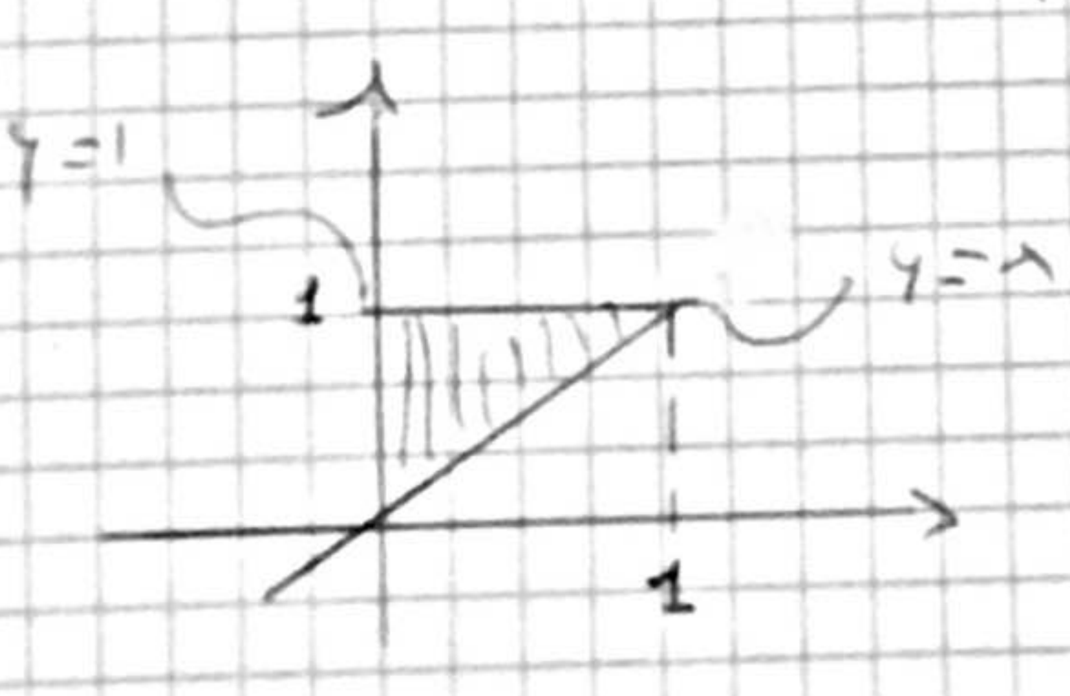
ESERCIZI SU INSIEMI NORMALI

- 

Normale rispetto a x

$D = \{ 0 \le x \le 1 \ ; \ 0 \le y \le x \}$

Normale rispetto a y $\quad D = \{ 0 \le y \le 1 \ ; \ y \le x \le 1 \}$

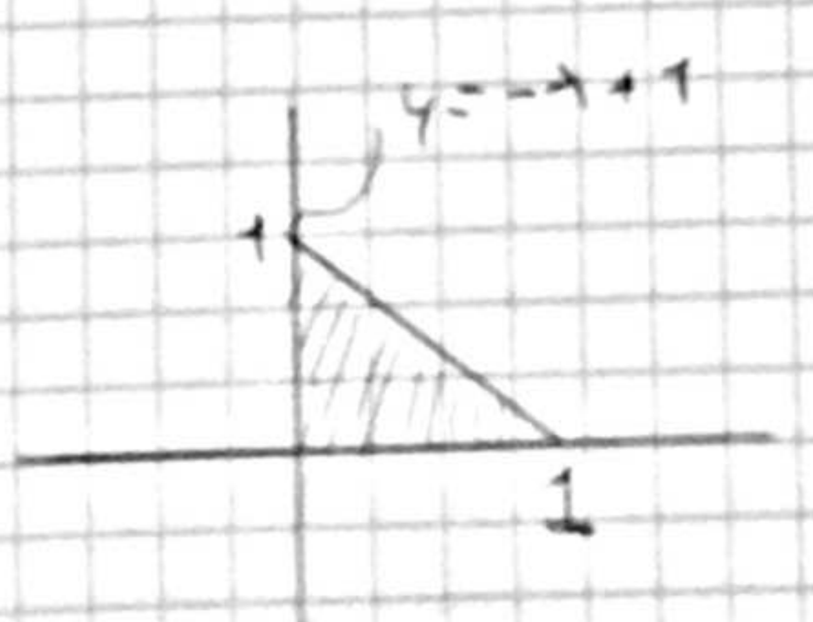
- 

Normale ad x

$D = \{ 0 \le x \le 1 \quad x \le y \le 1 \}$

Normale ad y

$D = \{ 0 \le y \le 1 \quad 0 \le x \le y \}$

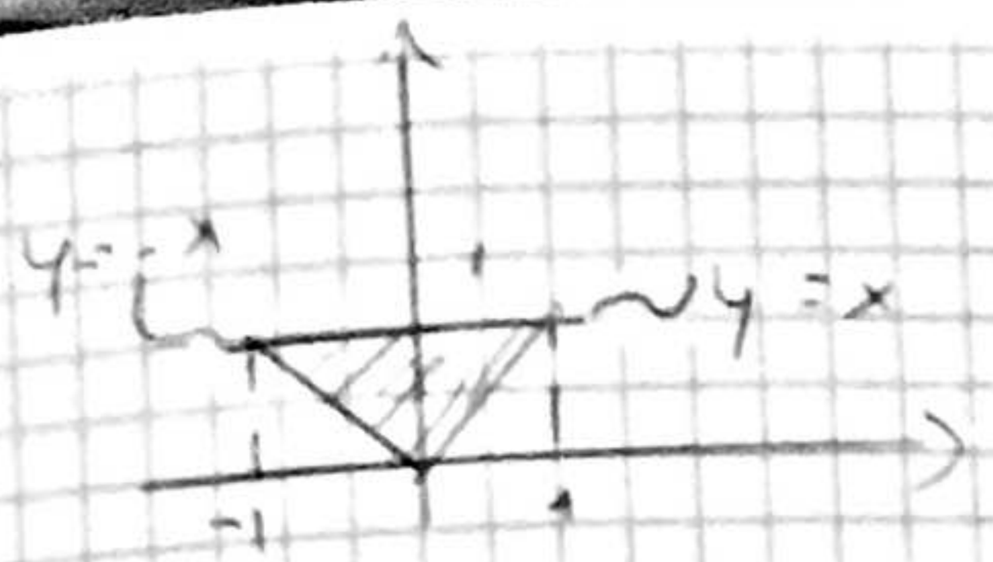
- 

Normale rispetto ad x

$D = \{ 0 \le x \le 1 \ ; \ 0 \le y \le 1 - x \}$

Normale rispetto ad y

$D = \{ 0 \le y \le 1 \quad 0 \le x \le 1 - y$



Insiemi normali ad x

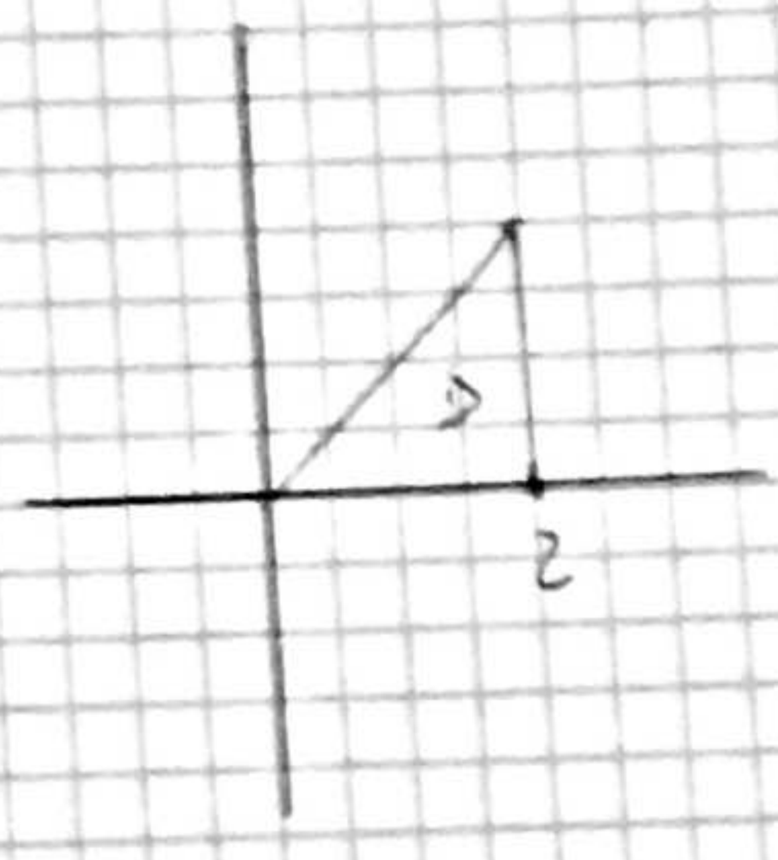
$$D = \{-1 \le x \le 1,\ |x| \le y \le 1\}$$

Insiemi normali ad y

$$D = \{0 \le y \le 1 \quad -y \le x \le y\}$$

INTEGRALI DOPPI ESERCIZI

$$D = \{x \in [0,2] \quad 0 \le y \le x\}$$



→ $f(x,y) = x + 6y$

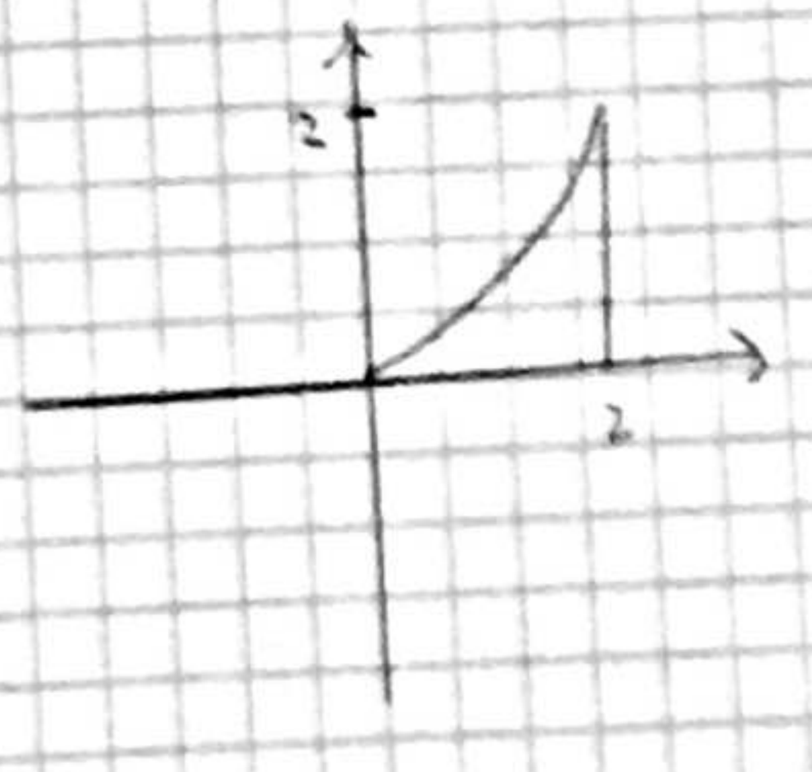
$$\int_0^2 dx \int_0^x x + 6y\, dy = \int_0^2 \left[xy + \frac{\cancel{6}^{3}}{\cancel{2}} y^2\right]_0^x =$$

$$= \int_0^2 dx\, (x^2 + 3x^2) = \frac{4}{3}x^3 \Big|_0^2 = \frac{32}{3}$$

→ $f(x,y) = x^3$

$$\int_0^2 dx \int_0^x x^3\, dy = \int_0^2 x^3 y \Big|_0^x dx = \frac{x^5}{5}\Big|_0^2 = \frac{32}{5}$$

• $D = \{x \in [0,2] \quad 0 \le y \le x^2\}$



$f(x,y) = xy$

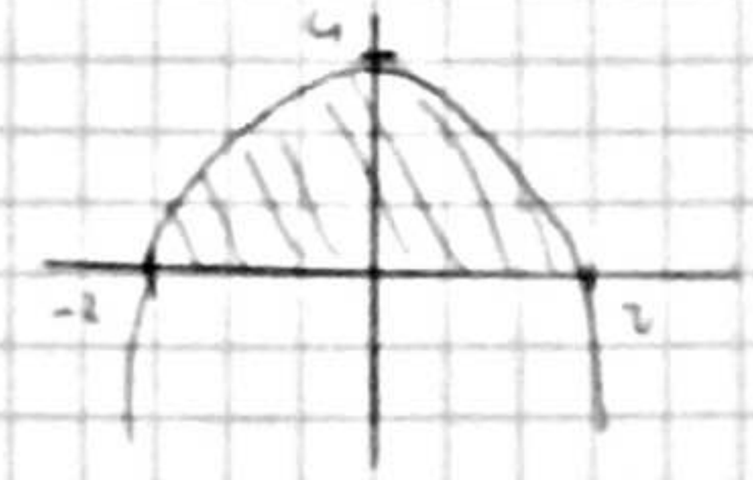
$$\int_0^2 dx \int_0^{x^2} xy\, dy = \int_0^2 x\frac{y^2}{2}\Big|_0^{x^2} =$$

$$= \int_0^2 \frac{x^5}{2} = \frac{x^6}{12}\Big|_0^2 = \frac{64}{12} = \frac{16}{3}$$

$$\int_0^2 dx \int_0^{x^2} x^3\, dy = \int_0^2 x^3 y \Big|_0^{x^2} dx = \int_0^2 x^5\, dx = \frac{64}{6} = \frac{32}{3}$$

• Sia $D = \{ 0 \le y \le 4 - x^2 \}$

$f(x,y) = x^2 y$ $\hookrightarrow -2 \le x \le 2$

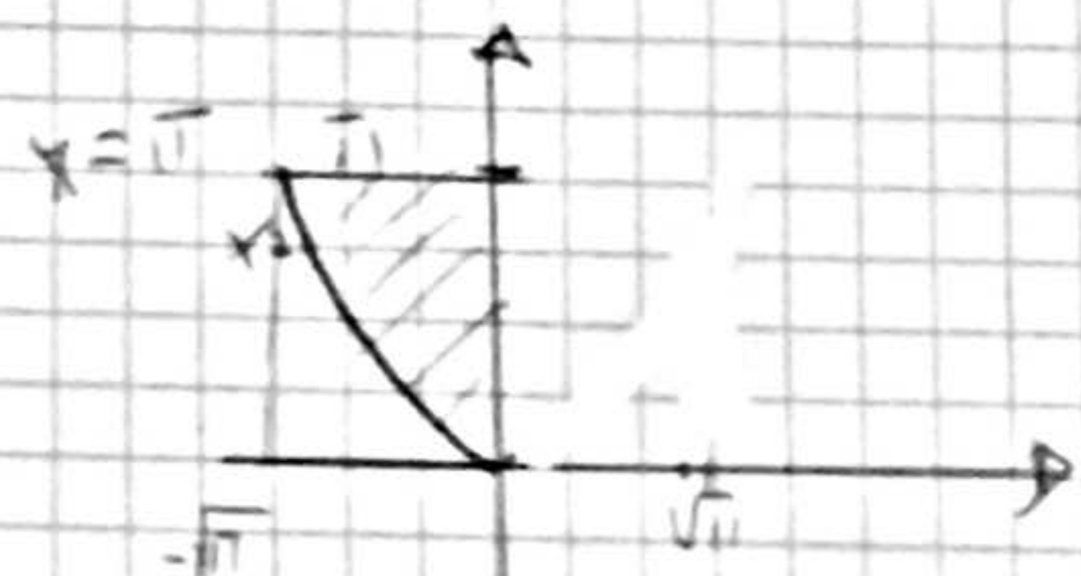


$\iint x^2 dy\, dx = \int_{-2}^{2} dx \int_0^{4-x^2} x^2 y\, dy = \int_{-2}^{2} x^2 \frac{y^2}{2}\Big|_0^{4-x^2} dx = \int_{-2}^{2} x^2 \frac{(4-x^2)^2}{2} dx =$

funzione pari $\rightarrow = \int_0^2 x^2 (4-x^2)^2 dx = \int_0^2 16x^2\, dx + \int_0^2 x^6\, dx - \int_0^2 8x^4\, dx =$

$= \frac{16}{3} x^3 \Big|_0^2 + \frac{1}{7} x^7 \Big|_0^2 - \frac{8}{5} x^5 \Big|_0^2 = \frac{16}{3} 8 + \frac{1}{7} 2^7 - \frac{8}{5} 2^5 = \frac{2^{10}}{105} \cdot \frac{1024}{105}$

• $D = \{ x \in [-\sqrt{\pi}, 0] \, , x^2 \le y \le \pi \}$



Chi è la primitiva di coseno?

$\int_{-\sqrt{\pi}}^{0} dx \int_{x^2}^{\pi} x \cos y = \int_{-\sqrt{\pi}}^{0} -x \sin y \Big|_{x^2}^{\pi} =$

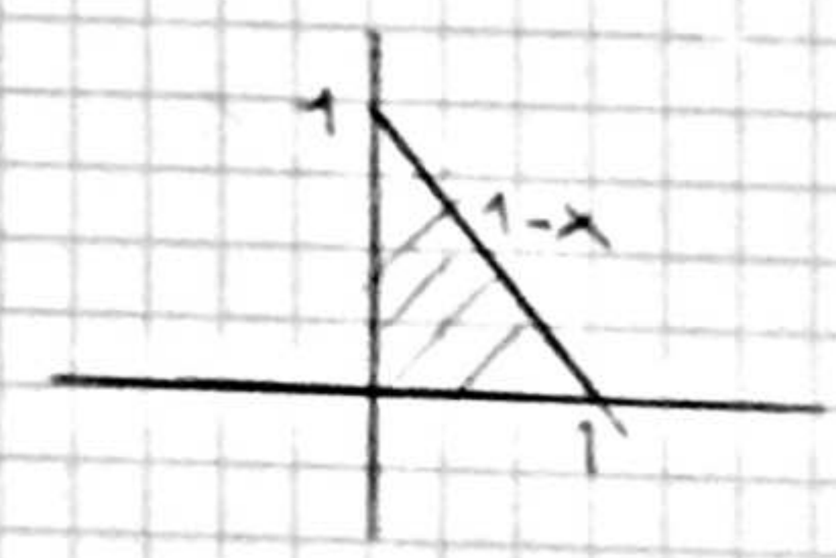
$= \int_{-\sqrt{\pi}}^{0} -x \sin x^2 dx = -\frac{1}{2} \int \sin u\, du = \frac{\cos(u)}{2} \Big|_{-\sqrt{\pi}}^{0} = \frac{\cos x^2}{2} \Big|_{-\sqrt{\pi}}^{0} = \frac{1}{2} + \frac{1}{2} = 1$

Segno sbagliato + segno sbagliato non fa segno giusto

$f(x,y) = x - 2y$

$D = \{ x \ge 0, y \ge 0 \; ; x + y \le 1 \}$

$0 \le x \le 1 \; ; \; 0 \le y \le 1 - x$



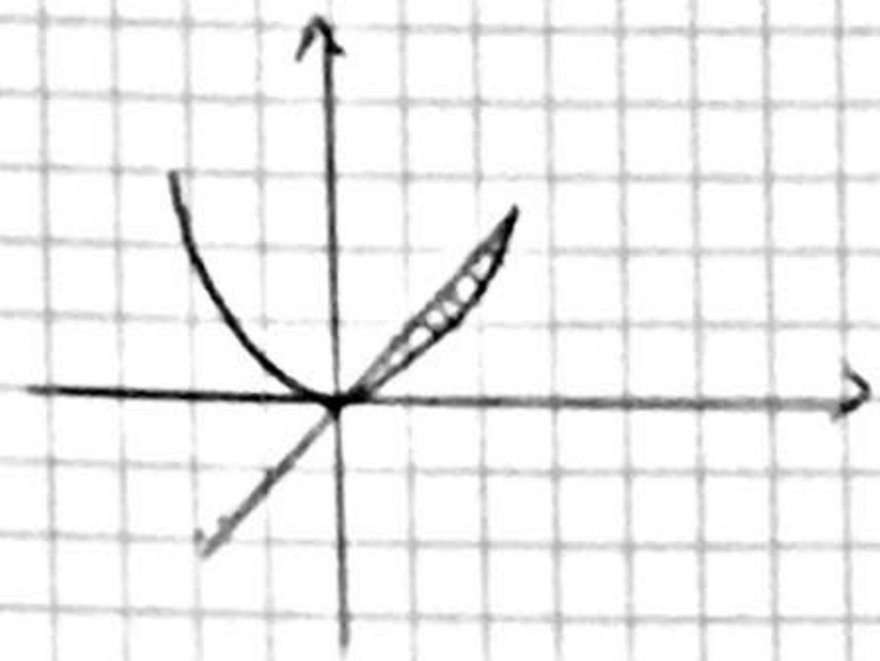
$\int_0^1 dx \int_0^{1-x} (x - 2y)\, dy = \int_0^1 dx \left[xy \Big|_0^{1-x} - 2 \left[\frac{y^2}{2} \right]_0^{1-x} \right] =$

$= \int_0^1 (x - x^2 - 1 - x^2 + 2x)\, dx = \int_0^1 (-2x^2 + 3x - 1)\, dx =$

$= \left[-\frac{2}{3} x^3 + \frac{3}{2} x^2 - x \right]_0^1 = -\frac{2}{3} + \frac{3}{2} - 1 = \frac{-4 + 9 - 6}{6} = -\frac{1}{6}$

• $f(x,y) = x \quad y$

$D = \{x^2 \leq y \leq x\}$



$$\int_0^1 dx \int_{x^2}^{x} (x^2 + xy^2)\, dy =$$

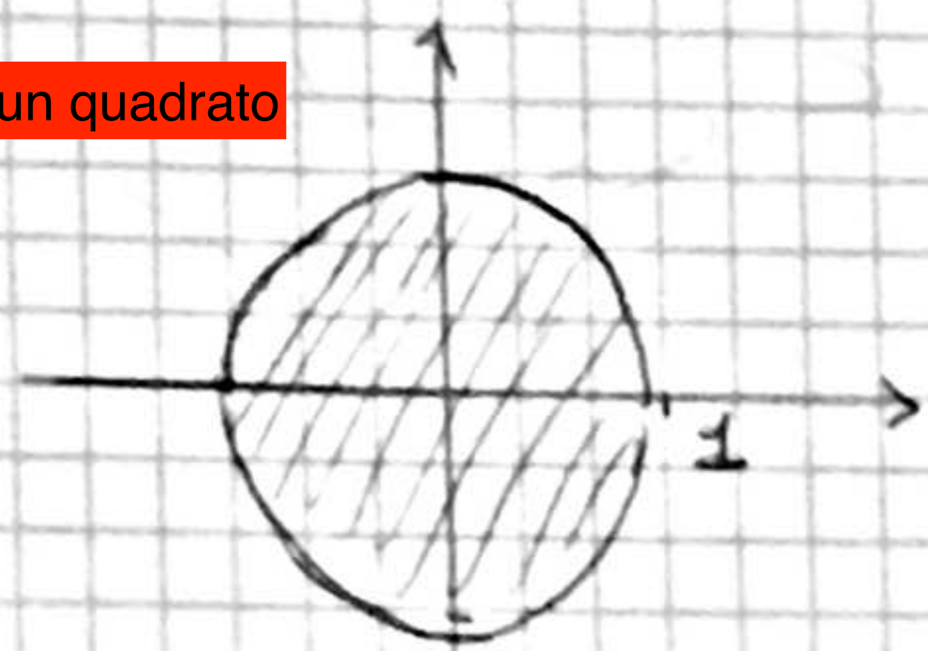
$$= \int_0^1 dx \left[x^2 y + x \frac{y^3}{3} \right|_{x^2}^{x} = \int_0^1 \left(x^3 - x^4 + \frac{x^4}{3} - \frac{x^7}{3} \right) dx =$$

$$= \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^5}{15} - \frac{x^8}{24} \Big|_0^1 = \frac{30 - 24 + 8 - 5}{120} = \frac{\cancel{9}^{3}}{\cancel{120}_{40}} = \frac{3}{40}$$

• $f(x,y) = x$ questo proprio no: lo hai scritto come un quadrato

$D = \{x^2 + y^2 \leq 1\}$

$\hookrightarrow \leq x \leq 1 \quad -1 \leq y \leq 1$



$$\int_{-1}^{1} dx \int_{-1}^{1} x\, dy = \int_{-1}^{1} (x + x)\, dx = \frac{x^2}{2} \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

- Sia ora $f(x,y) = y \rightarrow \int_{-1}^{1} dx \int_{-1}^{1} y\, dy = \int_{-1}^{1} \frac{y^2}{2} \Big|_{-1}^{1} dx = 0$

- Sia ora $f(x,y) = |x|$

Occhio gli estremi cambiano da sbagliati a giusti

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} x\, dy = \int_0^1 xy \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 2\int_0^1 x\sqrt{1-x^2} = -\frac{1}{3}(1-x^2)^{3/2} \Big|_0^1 = \frac{2}{3}$$

$$-\int_{-1}^{0} dx \int x\, dy \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = -2 \int_0^1 x\sqrt{1-x^2} = \frac{2}{3}(1-x^2)^{3/2} \Big|_{-1}^{0} = \frac{2}{3}$$

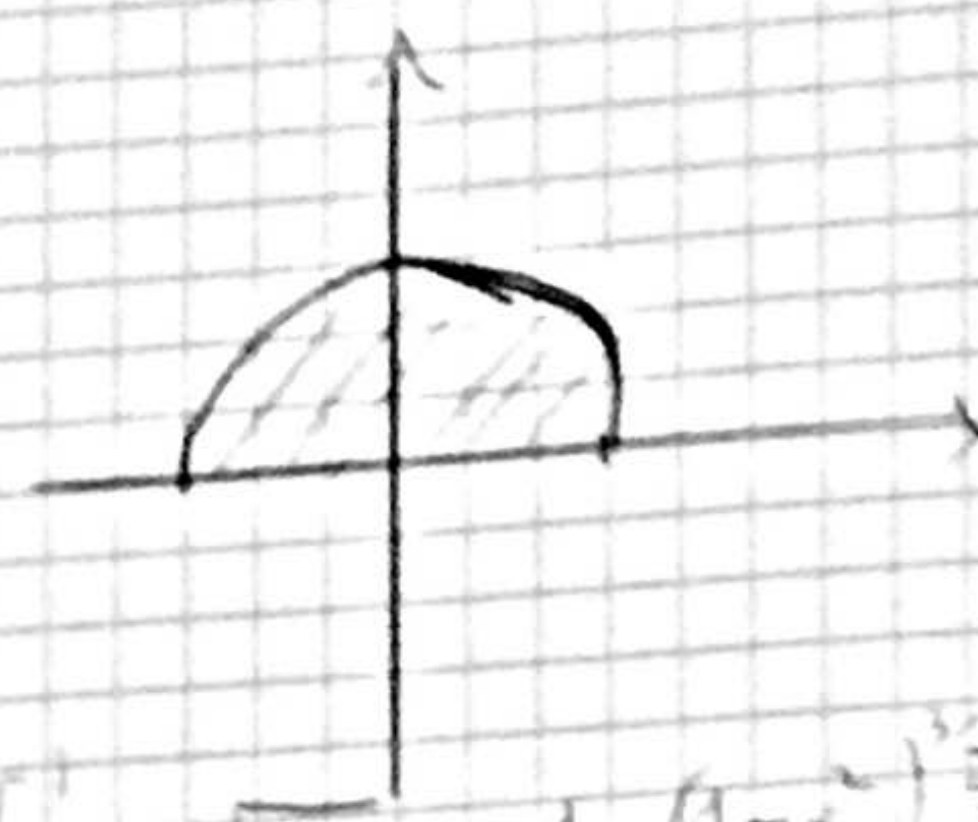
Somma le due parti e l'integrale è $\frac{4}{3}$

Via più semplice con le coordinate polari

ESERCIZIO 89

• $f(x,y) = x$

D lo scrivo come normale ad x



$D = \{-1 \le x \le 1;\ 0 \le y \le \sqrt{1-x^2}\}$

$$\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} x\, dy = \int_{-1}^{1} x y \Big|_{0}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} x \sqrt{1-x} = \frac{1}{3}(1-x^2)^{3/2} \Big|_{-1}^{1} = 0$$

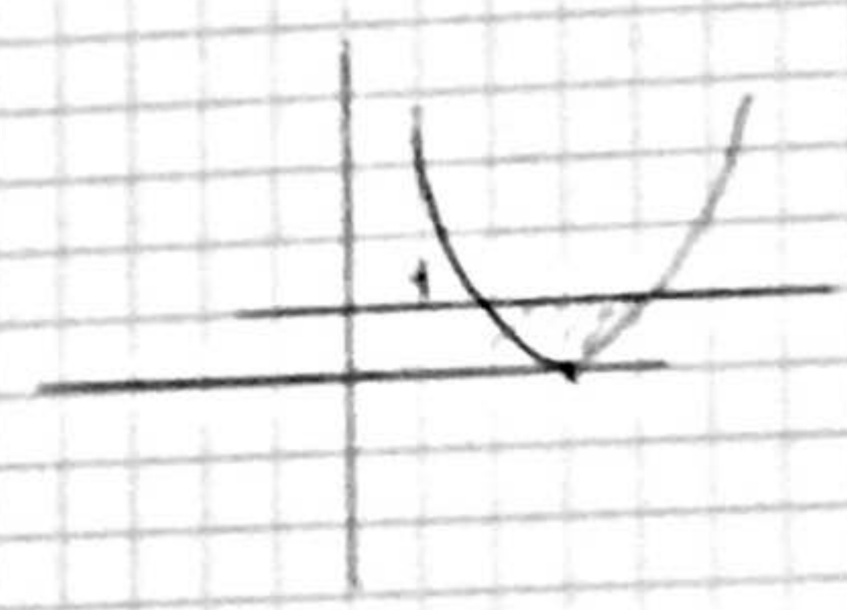
simmetrie!

$f(x,y) = y$

$$\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} y\, dy = \int_{-1}^{1} \frac{y^2}{2} \Big|_{0}^{\sqrt{1-x^2}} = \left[\frac{x}{2} - \frac{x^3}{6}\right]_{-1}^{1} = 1 - \frac{2}{6} = \frac{2}{3}$$

ESERCIZIO PAG 89

$f(x,y) = y^6 \qquad D = \{x^2-6x+9 \le y \le 1\}$



$$\begin{cases} y = 1 \\ y = x^2-6x+9 \end{cases}$$

$1 = x^2-6x+9$

$0 = x^2-6x+8$

$\Delta = 9-8 = 1$

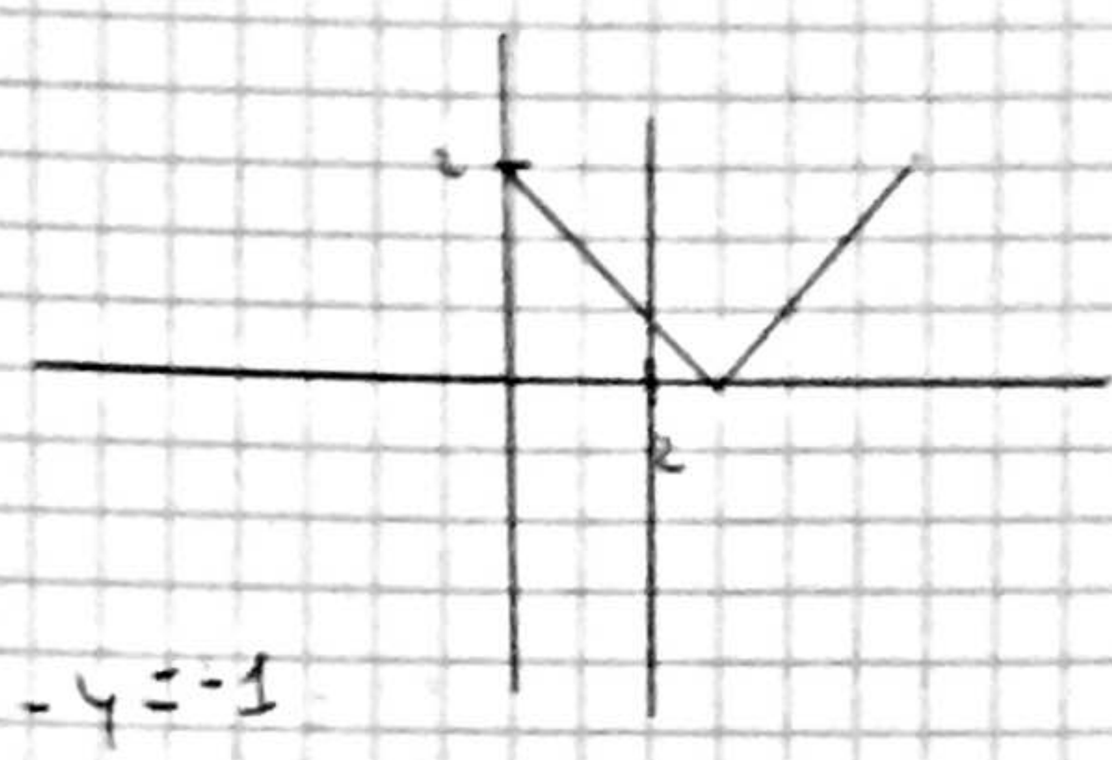
$x_{1,2} = +3 \pm \sqrt{1} = \begin{cases} 4 \\ 2 \end{cases}$

$$\int_{2}^{4} dx \int_{(x-3)^2}^{1} y^6\, dy = \int_{2}^{4} \left(\frac{1}{7} - \frac{(x-3)^{14}}{7}\right) dx = \frac{x}{7} - \frac{(x-3)^{15}}{15 \cdot 7} \Big|_{2}^{4} =$$

$$= \frac{4}{7} - \frac{1}{15 \cdot 7} - \frac{2}{7} - \frac{1}{15 \cdot 7} = \frac{2}{7} - \frac{2}{15 \cdot 7} = \frac{30-2}{15 \cdot 7} = \frac{28}{15 \cdot 7} = \frac{4}{15}$$

- $f(x,y) = y-3$

$D = \{ |y-3| \leq x \leq 2 \}$



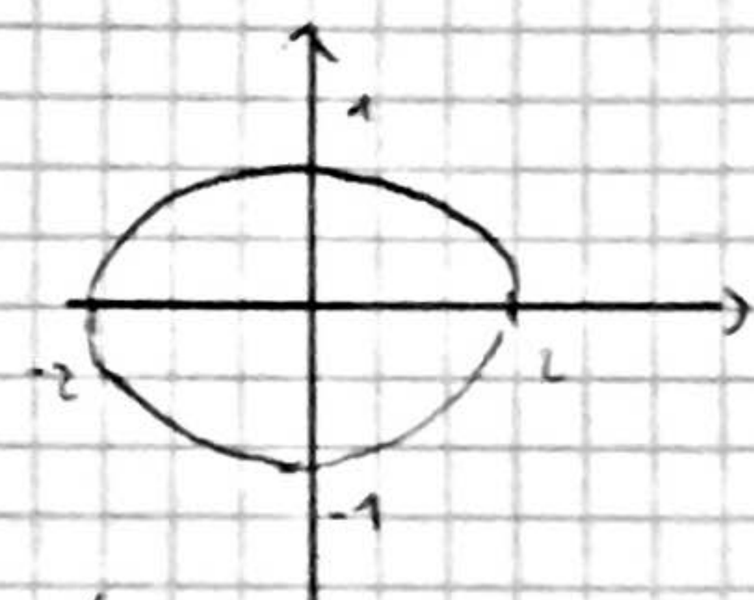
$$\begin{cases} x=2 \\ |y-3| = x \end{cases} \quad \begin{matrix} x = -y+3 \to -y = -1 \\ x = y-3 \to y=5 \end{matrix}$$

$$\int_0^2 dx \int_1^5 y-3 \, dy = \int_0^2 \frac{y^2}{2} - 3y \Big|_1^5 dx = \int_0^2 \left(\frac{25}{2} - 15 - \frac{1}{2} + 3\right) dx = 0$$

No, se fosse vero l'insieme sarebbe un rettangolo

- $f(x,y) = 1$

$D = \{ x^2 + 4y^2 \leq 4 \}$



$x = u$

$y = \frac{v}{2}$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \to \det J = \frac{1}{2}$$

$$\int_{u^2+v^2 \leq 4} \frac{1}{2} \, du \, dv = \frac{4\pi}{2} = 2\pi$$

SVILUPPARE FINO ALL'ORDINE 4

- $(1 + x^2 + y^2 + xy)^{1/4} - 1 = \cancel{1} + \frac{1}{4}(x^2+y^2+xy) + 1/4\,(1/4 - 1)\,[x^4 + y^4 + 2x^3y + 2xy^3$

$$+ x^2y^2 + 2x^2y^2 + o(x^2+y^2+xy)^2 - \cancel{1} =$$

$$= \frac{1}{4}(x^2+y^2+xy) + \left(-\frac{3}{32}\,[x^2+y^2+xy]^2\right) + o(x^2+y^2)^2$$

$$f_x = \frac{2x+y}{4(x^2+xy+y^2+1)^{3/4}} \to f_{xx} = \frac{1}{2(x^2+xy+1)^{3/4}} - \frac{3(2x+y)^2}{16\,[x^2+xy+y^2+1]^{7/4}}$$

$$f_y = \frac{x+2y}{4(x^2+xy+y^2+1)^{3/4}} \to f_{yy} = \frac{1}{2(x^2+xy+y^2+1)^{3/4}} - \frac{3(x+2y)^2}{16(x^2+xy+y^2+1)^{7/4}}$$

Una volta che hai gli sviluppi vicino a (0,0) non è così che si calcolano le derivate nell'origine! e non controllo che siano corrette….

$$f_{xy} = \frac{1}{4(x^2+xy+y^2+1)^{3/4}} - \frac{3(2x+y)(x+2y)}{16(x^2+xy+y^2+1)^{7/4}}$$

Devi anche dire che le derivate parziali sono nulle

$$H_f \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \Rightarrow H_f(0,0) \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix} \Rightarrow \det H_f(0,0) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

$h = 1$

è definita positiva $\Rightarrow$ (0,0) è min locale

- Fare lo sviluppo fino all'ordine 4° di

$\sin(x-y)\ \arctan(x-y)$

$\sin(x-y) = z$

Allora procedo con lo sviluppo di

$$\sin z\ \arctan(z) = \left[z - \frac{z^3}{6} + o(z^4)\right]\left[z - \frac{z^3}{3} + o(z^4)\right]$$

$$\to \sin(x-y)\arctan(x-y) = (x-y)^2 - \frac{(x-y)^4}{3} - \frac{(x-y)^4}{6} + o((x-y)^4)$$

$$= (x-y)^2 - \frac{(x-y)^4}{3} - \frac{(x-y)^4}{6} + o((x^2+y^2)^2) =$$

$$= (x-y)^2 - \frac{1}{2}(x-y)^4 + o((x^2+y^2)^2)$$

$$f_x = 1\cos(x-y)\arctan(x-y) + \sin(x-y)\frac{1}{(x-y)^2+1}$$

$$f_{xx} = -\frac{2(x-y)\sin(x-y)}{((x-y)^2+1)^2} + \frac{2\cos(x-y)}{(x-y)^2+1} - \sin(x-y)\tan^{-1}(x-y)$$

$$f_{xy} = \frac{2(x-y)\sin(x-y)}{((x-y)^2+1)^2} - \frac{2\cos(x-y)}{(x-y)^2+1} + \sin(x-y)\arctan(x-y)$$

Vedi sopra

$$f_y = -\cos(x-y)\arctan(x-y) - \frac{\sin(x-y)}{(x-y)^2+1}$$

$$f_{yy} = -\frac{2(x-y)\sin(x-y)}{((x-y)^2+1)^2} + \frac{2\cos(x-y)}{(x-y)^2+1} - \sin(x-y)\arctan(x-y)$$

$$H_f(0,0) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$\det H_f = 4+4 = 8$

$tr = 4$

No il determinante è nullo

la matrice è definita positiva allora ho min locale

• $\arctan(xy^3 + \sin(y))$ => fare lo sviluppo fino all'ordine 4

pongo $z = xy^3 + \sin y$

$\arctan z = z - \frac{z^3}{3} + o(z^4)$

$\to \arctan(xy^3+\sin y) = xy^3 + \sin y - \frac{(xy^3+\sin y)^3}{3} + o((xy^3+\sin y)^4)$

$= xy^3 + \sin y - \frac{\sin^3 y}{3} + o((xy^3+\sin y)^4)$

Una volta che hai sviluppato anche il seno non sono più questi gli o-piccoli corretti

$= xy^3 + y - \frac{y^3}{6} - \frac{y^3}{3} + o((xy^3+y)^4)$

$= xy^3 + y - \frac{y^3}{2} + o((x^2+y^2)^2)$

Poiché c'è il termine al primo ordine dello sviluppo allora il gradiente non è nullo quando calcolato nell'origine e quindi non è stazionario.

ESERCIZI INTEGRALI TRIPLI

• $D = \{[0,1]\times[0,2]\times[0,3]\}$

$f(x,y,z) = x$ $\to \int_0^1 dx \int_0^2 dy \int_0^3 x\,dz = \int_0^1 dx \int_0^2 3x\,dy = \int_0^1 6x\,dy$

$= 3$

• $D = \{[0,\pi]\times[0,\pi]\times[0,\pi]\}$

$f(x,y,z) = \sin y$ $\to \int_0^\pi dx \int_0^\pi dy \int_0^\pi \sin y\,dz =$

$= \int_0^\pi dx \int_0^\pi \pi \sin y\,dy = -\int_0^\pi \pi\cos y\Big|_0^\pi dx =$

$= 2\pi \int_0^\pi dx = 2\pi^2$

• $D = \{(x,y) \in [0,1]\times[0,1],\ 0 \le z \le xy\}$

$f(x,y,z) = y$

$\int_0^1 dx \int_0^1 dy \int_0^{xy} y\,dt = \int_0^1 dx \int_0^1 y^2 x\,dy =$

$= \int_0^1 \frac{y^3}{3}x\Big|_0^1 dx = \int_0^1 \frac{x}{3}\,dx = \frac{x^2}{6}\Big|_0^1 = \frac{1}{6}$

$\det J = \begin{pmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{pmatrix}$

uso le polari

- $D = \{ x^2+y^2 \le 4 \ , z \in [-1,2] \}$

$f(x,y,z) = x^2$

$$\int_{x^2+y^2\le 4} dx\,dy \int_{-1}^{2} x^2\,dz = \int_{x^2+y^2\le 4} x^2 z \Big|_{-1}^{2} dx\,dy = \int_{x^2+y^2\le 4} 3x^2\,dx\,dy =$$

$$= 3\int_{0}^{2} d\rho \int_{0}^{2\pi} \rho\cdot\rho^2\cos^2\theta\,d\theta = 3\pi \int_{-2}^{2} \rho^3\,d\theta = 3\pi \frac{\rho^4}{4}\Big|_{0}^{2} = 12\pi$$

- $D = \{ x^2+y^2 \le 4 \ ; \ z \in [-1,2] \}$

$f(x,y,z) = x^2+y^2+z^2$

$$\int_{x^2+y^2\le 4} dx\,dy \int_{-1}^{2} x^2+y^2+z^2\,dz = \int_{x^2+y^2\le 4} \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_{-1}^{2} dx\,dy =$$

$$= \int_{x^2+y^2\le 4} (3x^2+3y^2+1)\,dx\,dy =$$

spezzo nella somma di 2 integrali

qui c'è un 3

$$= 3\int_{0}^{2\pi} d\theta \int_{0}^{2} \rho\cdot\rho^2\,d\rho + \int_{x^2+y^2\le 4} 3\,dx\,dy = 24\pi + 12\pi = 36$$

viene dalle coordinate polari dove

$\det J = \begin{pmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{pmatrix}$

area cerchio di raggio 2

• $D = \{ x^2+y^2+z^2 \leq 4 \}$ $\Rightarrow$ Lo scrivo come normale rispetto a z

↳ sfera di raggio 2

$f(x,y,z) = z-1$

$\bar{D} = \{ -2 \leq z \leq 2 \ ; \ x^2+y^2 \leq 1-z^2 \}$

$$\int_{x^2+y^2+z^2 \leq 4} z-1 \, dx\,dy\,dz = \int_{-2}^{2} dz \int_{x^2+y^2 \leq 4-z^2} z-1 \, dx\,d\underset{\substack{\uparrow \\ 1}}{y}$$

area cerchio raggio $\sqrt{4-z^2}$

$$= \pi \int_{-2}^{2} dz \, (4-z^2)(z-1)$$

$$= \pi \int_{-2}^{2} dz \left[4z - 4 - z^3 + z^2 \right] =$$

$$= \pi \left[\overset{2}{\cancel{4}} \frac{z^2}{\cancel{2}} - 4z - \frac{z^4}{4} + \frac{z^2}{3} \right]_{-2}^{2} = \pi \left[8 - 8 - \cancel{4} + \frac{8}{3} - \left(8 + 8 - \cancel{4} - \frac{8}{3} \right) \right]$$

$$= \pi \left[\frac{16}{3} - 16 \right] = -\frac{32}{3}$$