Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 03 September 2019

1. Determine whether the functional
$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u}u + u^2 + u\right)\, dx$$
has the minimum in the class $C^1([0,1])$.

2. Let us consider the boundary value problem
$$u'' = -\frac{x^3}{u^3}, \qquad u(0) = 1, \quad u(2) = 3.$$
   (a) Discuss existence, uniqueness and regularity of positive solutions.
   (b) Determine the minimum of the solution in the interval $[0,2]$.

3. Let us consider, for every $\ell > 0$, the problem
$$\inf\left\{\int_0^\ell \arctan\left(\dot{u}^2 - u^2\right)\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$
   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.
   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.
   (c) Determine the infimum as a function of ℓ.

4. For every real number $m > 0$, let us set
$$J(m) := \inf\left\{\int_0^1 \left(u^{19} - \sin(u^2)\right)\, dx : u \in C^1_c((0,1)),\ \int_0^1 |\dot{u}|^7\, dx \leq m\right\}.$$
   (a) Determine whether there exists $m > 0$ such that $J(m) = 0$.
   (b) Determine for which real values of α it turns out that
$$\lim_{m\to 0^+} \frac{J(m)}{m^\alpha} = 0.$$
   (c) Determine for which real values of β it turns out that
$$\lim_{m\to +\infty} \frac{J(m)}{m^\beta} = 0.$$

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.