ESERCIZIO 1 — 90 numeri estratti uno alla volta senza reinserimento

X = primo numero estratto

Y = secondo " "

a) X e Y assumono valori in $\{1,2,3,\dots,90\}$

$$P[X=k] = \frac{1}{90} \quad \forall k \in \{1,\dots,90\}$$

$$P[Y=k] \underset{h\neq k}{=} \frac{P[X=h \text{ e } Y=k]}{P[X=h \mid Y=k]} = \frac{\frac{1}{90\cdot 89}}{\frac{1}{89}} = \frac{1}{90} \quad \forall k \in \{1,\dots,90\}$$

X e Y sono indipendenti $\iff \forall A, B \subset \Omega$ vale che

$$P(X\in A \cap Y\in B) = P(X\in A)\,P(Y\in B)$$

Poiché $P(X=k \text{ e } Y=k) = 0 \neq P(X=k)\cdot P(Y=k)$

$P(X=k \text{ e } Y=h) = \frac{1}{90\cdot 89} \neq P(X=k)\cdot P(Y=h)$

X e Y non sono indipendenti

b) $E[X+Y] = E[X] + E[Y] = 2\,E[X] =$

$$= 2 \sum_{k=1}^{90} k \cdot P(X=k) = 2\cdot \sum_{k=1}^{90} k\cdot \frac{1}{90} = \frac{2}{90}\sum_{k=1}^{90} k =$$

$$= \frac{2}{90}\cdot \frac{91\cdot 90}{2} = 91 \qquad \left(\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right)$$

c) Estraendo 10 numeri, qual è la prob. di ottenere 20?

$P(A) = \dfrac{\#A}{\#\Omega}$ — $\Omega = \{$decine estratte$\}$

$A = \{$decine che contengono il 20$\}$

$$\# \Omega = \binom{90}{10} \qquad \# A = \binom{89}{9}$$

$$P(A) = \frac{\binom{89}{9}}{\binom{90}{10}} = \frac{89!}{9!\,80!} \cdot \frac{10!\,80!}{90!} = \frac{10}{90} = \frac{1}{9}$$

ESERCIZIO 2 Sia X v.a. con densità

$$f(x) = \begin{cases} x\, e^{-x} & , \quad x > 0 \\ 0 & , \quad x \leq 0 \end{cases}$$

$X \sim \Gamma(2,1)$, e poniamo $Y = X^{-1}$.

a) $E[XY]$ $\qquad (XY)(\omega) = X(\omega) Y(\omega) = 1$

$$E[XY] = E[g(X)] = \int_{\mathbb{R}} g(x) f(x)\, dx = \int_{\mathbb{R}} f(x)\, dx = 1$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 1 - 2 = -1$$

$$E[X] = 2$$

$$E[Y] = E\left[\frac{1}{X}\right] = \int_{\mathbb{R}} \frac{1}{x} f(x)\, dx = \int_0^{+\infty} e^{-x}\, dx = 1$$

b) Quali momenti possiede Y?

Y ha momento $k \in \mathbb{N}$ se $E[Y^k] < +\infty$.

$$E[Y^k] = \int_{\mathbb{R}} \left(\frac{1}{x}\right)^k f(x)\, dx = \int_0^{+\infty} x^{1-k} e^{-x}\, dx =$$

$$= \int_0^1 x^{1-k} e^{-x}\, dx + \underbrace{\int_1^{+\infty} x^{1-k} e^{-x}\, dx}_{< +\infty \ \forall k} < +\infty \iff 1-k > -1$$

$$\iff k < 2$$

$$\int_0^1 x^{\alpha}\, dx < +\infty \iff \alpha > -1$$

Quindi Y possiede solo il momento primo.

c) $Y = X^{-1} = h(X)$ $\quad h(t) = t^{-1}$

h invertibile su $(0, +\infty)$

$$f_Y(y) = \begin{cases} f_X(h^{-1}(y)) \left| \frac{d h^{-1}}{dy}(y) \right| , & y \in h(0,+\infty) \Leftrightarrow y > 0 \\ 0 , & y \le 0 \end{cases}$$

$$h^{-1}(y) = \frac{1}{y} , \quad \left| \frac{d h^{-1}}{dy}(y) \right| = \frac{1}{y^2}$$

$$f_Y(y) = \begin{cases} \frac{1}{y} e^{-\frac{1}{y}} \cdot \frac{1}{y^2} = \frac{1}{y^3} e^{-\frac{1}{y}} , & y > 0 \\ 0 , & y \le 0 \end{cases}$$

d) r_β sia il β-quantile di X, r_β è l'unica soluzione di

$$\beta = F_X(r_\beta) \quad \forall \beta \in (0,1)$$

s_β sia il β-quantile di Y, si ha $\beta = F_Y(s_\beta)$

Si ha $r_\beta, s_\beta \in (0, +\infty) \quad \forall \beta \in (0,1)$

$$\beta = F_Y(s_\beta) = P[Y \le s_\beta] = P\left[\frac{1}{X} \le s_\beta\right] = P\left[\frac{1}{s_\beta} \le X\right] =$$

$$= 1 - P\left[X \le \frac{1}{s_\beta}\right] = 1 - F_X\left(\frac{1}{s_\beta}\right)$$

$\Leftrightarrow \quad F_X\left(\frac{1}{s_\beta}\right) = 1 - \beta = \tilde{\beta} \quad$ allora $\frac{1}{s_\beta} = r_{\tilde{\beta}}$

Quindi $\boxed{s_\beta = \frac{1}{r_{1-\beta}}}$.

ESERCIZIO 3   Per $\vartheta \in (0,+\infty)$ sia

$$f_\vartheta(x) = \begin{cases} c_\vartheta \, x^{-(1+\vartheta)}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

a) $f_\vartheta \geq 0$, f_ϑ integrabile, $\int_{\mathbb{R}} f_\vartheta(x)\,dx = 1$

$\Downarrow$

$c_\vartheta \geq 0$, e , $1 = \int_1^{+\infty} c_\vartheta \, x^{-(1+\vartheta)}\,dx = c_\vartheta \frac{x^{-\vartheta}}{-\vartheta}\Big|_1^{+\infty} = \frac{c_\vartheta}{\vartheta}$

$\Rightarrow \quad c_\vartheta = \vartheta$

$$F_X(x) = \int_{-\infty}^x f_\vartheta(t)\,dt = \begin{cases} 0, & x < 1 \\ 1 - x^{-\vartheta}, & x \geq 1 \end{cases}$$

$$\int_1^x f_\vartheta(t)\,dt = \int_1^x \vartheta \, t^{-(1+\vartheta)}\,dt = \vartheta \frac{t^{-\vartheta}}{-\vartheta}\Big|_1^x = 1 - x^{-\vartheta}$$

b) Per quali ϑ $E[X^k] < +\infty$ per $k = 1, 2$?

$$E[X^k] < +\infty \iff \int_{\mathbb{R}} x^k f_\vartheta(x)\,dx < +\infty \iff \vartheta > k$$

$$\int_1^{+\infty} x^k \, \vartheta \, x^{-(1+\vartheta)}\,dx = \int_1^{+\infty} \vartheta \, x^{k-1-\vartheta}\,dx < +\infty \iff k-1-\vartheta < -1$$

$$\int_1^{+\infty} x^\beta\,dx < +\infty \iff \beta < -1$$

Quindi $E[X]$ ed $E[X^2]$ sono finiti se $\vartheta > 2$.

c) Date $X_1, \dots, X_n$ con distribuzione f_ϑ, chiamiamo $x_1, \dots, x_n$ dati.

Metodo di massima verosimiglianza.

$$L_\vartheta(x_1,\dots,x_n) = \prod_{j=1}^{n} f_\vartheta(x_j) = \begin{cases} 0 , & \text{se } \exists\, x_j \in (-\infty,1) \\ \vartheta^n \, (x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{-(1+\vartheta)} , & \text{se } x_j \ge 1 \ \forall j \end{cases}$$

$\vartheta \in (0,+\infty)$

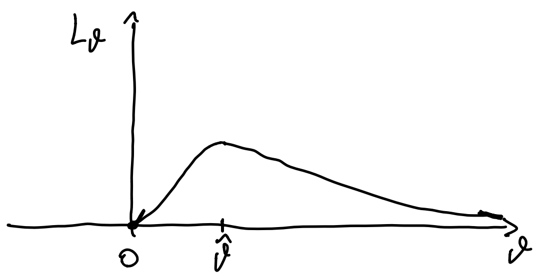
$L_\vartheta(x_1,\dots,x_n) > 0$ e quindi $\frac{d}{d\vartheta} L_\vartheta(x_1,\dots,x_n) = 0 \iff \frac{d}{d\vartheta} \log L_\vartheta(x_1,\dots,x_n) = 0$

$$\frac{d}{d\vartheta} \log L_\vartheta(x_1,\dots,x_n) = \frac{d}{d\vartheta}\Big(n \log\vartheta - (1+\vartheta)\log(x_1\cdot\ldots\cdot x_n)\Big) =$$

$$= \frac{n}{\vartheta} - \log(x_1\cdot\ldots\cdot x_n)$$

$$\frac{d}{d\vartheta} L_\vartheta(x_1,\dots,x_n) = 0 \iff \hat\vartheta = \frac{n}{\log(x_1\cdot\ldots\cdot x_n)}$$

$\hat\vartheta$ è un buon valore se $\hat\vartheta > 0$, è vero se $\exists\, x_j > 1$.



Se $x_1 = \ldots = x_n = 1$, allora $L_\vartheta(1,\dots,1) = \vartheta^n$

$\nexists\ \hat\vartheta$

Metodo dei momenti

$$E[X] = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$\int_{\mathbb{R}} x\, f_\vartheta(x)\, dx = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$\vartheta > 1$ $$\int_1^{+\infty} \vartheta\, x^{-\vartheta}\, dx = \vartheta\, \frac{x^{1-\vartheta}}{1-\vartheta}\Big|_1^{+\infty} = \frac{\vartheta}{\vartheta-1}$$

$\tilde{\vartheta}$ è la soluzione di $\frac{\vartheta}{\vartheta-1} = \frac{1}{n}\sum_{j=1}^{n} x_j$

$$\tilde{\vartheta} = \frac{\frac{1}{n}\sum_{j=1}^{n} x_j}{\frac{1}{n}\sum_{j=1}^{n} x_j - 1}$$

Ha senso se $\tilde{\vartheta} > 1$, è vero se $\exists\, x_j > 1$.

Se invece $x_1 = \dots = x_n = 1$, allora $\frac{1}{n}\sum_{j=1}^{n} x_j = 1$ e $\nexists$ soluzione di $\frac{\vartheta}{\vartheta-1} = 1$ per $\vartheta > 1$.

Passo al momento secondo, $E[X^2] = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = 1$

$\exists\, \vartheta > 2$ t.c. $\int_{\mathbb{R}} x^2 f_\vartheta(x)\, dx = 1$?

$$\int_1^{+\infty} \vartheta\, x^{1-\vartheta}\, dx = \vartheta\, \frac{x^{2-\vartheta}}{2-\vartheta}\Big|_1^{+\infty} = \frac{\vartheta}{\vartheta-2} = 1 \ ?$$ $\nexists$ soluz.

In generale

Se $\vartheta > k$, $$E[X^k] = \int_{\mathbb{R}} x^k f_\vartheta(x)\, dx = \int_1^{+\infty} \vartheta\, x^{k-1-\vartheta}\, dx = \vartheta\, \frac{x^{k-\vartheta}}{k-\vartheta}\Big|_1^{+\infty} =$$
$$= \frac{\vartheta}{\vartheta-k} = 1$$ non ha soluzione.

ESERCIZIO 3 Vecchio programma

Catena di Markov su $S = \{1, 2, 3, 4, 5\}$.

a)

$$P = \left(\begin{array}{ccc|cc} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/3 & 2/3 \end{array}\right)$$

b) $P(X_2 = 3 \mid X_0 = 1) = \frac{1}{9} + \frac{1}{3} = \frac{4}{9}$

$$(1\ 0\ 0\ 0\ 0)\, P^2 = \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ 0\ 0\right) P =$$

$$= \left(*\ \ *\ \ \tfrac{1}{9} + \tfrac{1}{6} + \tfrac{1}{6}\ \ 0\ \ 0\right)$$

Se $X_0 = 4$?

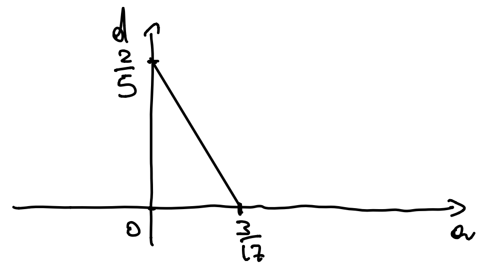
$$(0\ 0\ 0\ 1\ 0)\, P^2 = \left(0\ 0\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\right) P =$$

$$= \left(0\ \ 0\ \ 0\ \ \tfrac{1}{4} + \tfrac{1}{6}\ \ \tfrac{1}{4} + \tfrac{1}{3}\right)$$

c) $(a\ b\ c\ d\ e)\, P = (a\ b\ c\ d\ e)$, $\quad a+b+c+d+e = 1$

$$\begin{cases} \frac{1}{3}a + \frac{1}{4}c = a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{4}c = b \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}c = c \\ \frac{1}{2}d + \frac{1}{3}e = d \\ \frac{1}{2}d + \frac{2}{3}e = e \end{cases} \Rightarrow \left(a\ \ 2a\ \ \tfrac{8}{3}a\ \ d\ \ \tfrac{3}{2}d\right)$$

$$\frac{17}{3}a + \frac{5}{2}d = 1, \quad a \geq 0, \ d \geq 0$$


d
$\frac{2}{5}$
0
$\frac{3}{17}$
a