<u>ES 1</u> Album di 10 figurine.

1 figurina inserita

X = # tentativi per trovare la seconda figurina.

Y = # tentativi " " " terza figurina.

a) X assume valori in $\mathbb{N}$

$$P\{X=1\} = \frac{9}{10} \quad , \quad P\{X=2\} = \frac{1}{10}\cdot\frac{9}{10}$$

X è v.a. geometrica di parametro $p = \frac{9}{10}$

$$P\{X=k\} = (1-p)^{k-1}p = \frac{1}{10^{k-1}}\cdot\frac{9}{10}$$

b) Y assume valori in $\mathbb{N}$

$$P\{Y=k\} = (1-q)^{k-1}q = \frac{1}{5^{k-1}}\cdot\frac{4}{5}$$

c)
$$P\{X+Y=3\} = \sum_{k=0}^{3} P\{X=k\}\,P\{Y=3-k\}$$

$$= \sum_{k=1}^{2} P\{X=k\}\,P\{Y=3-k\} =$$

$$= P\{X=1\}\,P\{Y=2\} + P\{X=2\}\,P\{Y=1\}$$

$$= \frac{9}{10}\cdot\frac{4}{25} + \frac{9}{100}\cdot\frac{4}{5} = \frac{27}{125}$$

ES. 2 $\quad F(x) = \begin{cases} 0, & x < 1 \\ 1 - \frac{1}{x^3}, & x \geq 1 \end{cases}$

a) F è continua, debolmente crescente e $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

$f(x) = F'(x)$ dove F è derivabile

$$f_X(x) = \begin{cases} 0, & x < 1 \\ 3 \frac{1}{x^4}, & x > 1 \end{cases}$$

$$E[X^k] = \int_{-\infty}^{+\infty} x^k f_X(x)\, dx = \int_1^{+\infty} 3\, x^{k-4}\, dx \quad < +\infty$$

se e solo se $k-4 < -1 \iff k < 3$

$$E[X] = \int_1^{+\infty} 3\, x^{-3}\, dx = \frac{3}{2}$$

$$E[X^2] = \int_1^{+\infty} 3\, x^{-2}\, dx = 3$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{3}{4}$$

$E[X^3]$ non esiste

b) $X, Y$ v.a. indip ed equidistribuite

$$E[(X-Y)^2] = Var(X-Y) + E[X-Y]^2$$

$$E[X-Y] = E[X] - E[Y] = 0$$

$$\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$
$$= 2\,\mathrm{Var}(X)$$

$$E[(X-Y)^2] = \frac{3}{2}$$

c) $X_1, \dots, X_{80}$ indip ed equidist.

$$P\{X_1 + \dots + X_{80} \leq 100\} = P\left\{\frac{X_1 + \dots + X_{80} - 120}{\sqrt{60}} \leq \frac{100-120}{\sqrt{60}}\right\}$$

$$\frac{X_1 + \dots + X_{80} - 80\,E[X_i]}{\sqrt{80\,\mathrm{Var}(X_i)}} \sim N(0,1) = Z$$

$$80\,E[X_i] = 120$$
$$80\,\mathrm{Var}(X_i) = 60$$

$$\sim P\left\{Z \leq -\frac{20}{\sqrt{60}}\right\} = \Phi\left(-\frac{10}{\sqrt{15}}\right) \left(= 0.00491 \quad R\right)$$

$$\sim \Phi(-2.58199) = 1 - \Phi(2.58199) \sim 1 - 0.995$$
$$= 0.005$$

ES 3 $\quad X_1, \dots, X_{142} \quad X_k \sim B(1, p) \quad \hat{p} = 0.23$

a) Intervallo di fiducia bilatero per $p$

$$I = \left[ \hat{p} - q_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \, , \, \hat{p} + q_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]$$

$1-\alpha = 0.9 \iff \alpha = 0.1 \iff 1-\alpha/2 = 0.95$

$n = 192 \qquad \hat{p} = 0.23$

Precisione della stima

$$q_{0.95} \sim 1.645 \quad (= 1.644854 \ R)$$

$$q_{0.95} \sqrt{\frac{0.23(1-0.23)}{192}} \sim 0.0499$$

b) $H_0)\ p \geq 0.25 = p_0 \qquad H_1)\ p < 0.25$

Test unilatero per $p$ al livello $\alpha$

Regione critica $\quad C = \left\{ \frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}} (\hat{p} - p_0) < q_\alpha \right\}$

p-valore $\quad \bar{\alpha} = \Phi\left( \frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}} (\hat{p} - p_0) \right) = \Phi(-0.64)$

$(= 0.26108 \ R)$

$= 1 - \Phi(0.64) \sim 1 - 0.73891 = 0.26109$

ES.3 (vecchio programma)

Catena di Markov su $S = \{1, 2, 3, 4\}$

$$P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

a) $X = (a \quad b \quad c \quad d)$ t.c. $a, b, c, d \geq 0$

$$\begin{cases} a+b+c+d = 1 \\ X P = X \end{cases}$$

$$\begin{cases} a+b+c+d = 1 \\ \frac{1}{4}a + \frac{1}{4}b + \frac{1}{2}c = a \\ \frac{1}{2}b + \frac{1}{3}d = b \\ \frac{3}{4}a + \frac{1}{4}b + \frac{1}{2}c + \frac{1}{3}d = c \\ \frac{1}{3}d = d \end{cases} \iff \begin{cases} a+c = 1 \\ \frac{1}{4}a + \frac{1}{2}c = a \\ b = 0 \\ \frac{3}{4}a + \frac{1}{2}c = c \\ d = 0 \end{cases}$$

$$\iff \quad a = \frac{2}{5}, \quad c = \frac{3}{5}$$

$$X = \left( \frac{2}{5} \quad 0 \quad \frac{3}{5} \quad 0 \right)$$

b) $X_0 = (0 \quad 0 \quad 0 \quad 1)$

$$P\{X_2 = 3\} = \sum_{j=1}^{4} \overbrace{P\{X_2 = 3 \mid X_1 = j\}}^{= P_{j3}} P\{X_1 = j\}$$

$$X_1 = X_0 P = \left(0 \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}\right)$$

$$P\{X_2 = 3\} = \frac{1}{12} + \frac{1}{6} + \frac{1}{9} = \frac{13}{36}$$