

ES 1 Album di 10 figurine.

1 figurina mancante

X = # tentativi per trovare la seconda figurina.

Y = # tentativi " " " terza figurina.

a) X assume valori in \mathbb{N}

$$P\{X=1\} = \frac{9}{10}, \quad P\{X=2\} = \frac{1}{10} \cdot \frac{9}{10}$$

X è v.e. geometrica di parametro $p = \frac{9}{10}$

$$P\{X=k\} = (1-p)^{k-1} p = \frac{1}{10^{k-1}} \cdot \frac{9}{10}$$

b) Y assume valori in \mathbb{N}

$$P\{Y=k\} = (1-q)^{k-1} q = \frac{1}{5^{k-1}} \cdot \frac{4}{5}$$

$$c) \quad P\{X+Y=3\} = \sum_{k=0}^3 P\{X=k\} P\{Y=3-k\}$$

$$= \sum_{k=1}^2 P\{X=k\} P\{Y=3-k\}$$

$$= P\{X=1\} P\{Y=2\} + P\{X=2\} P\{Y=1\}$$

$$= \frac{9}{10} \cdot \frac{4}{25} + \frac{9}{100} \cdot \frac{4}{5} = \frac{27}{125}$$



ES. 2
$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - \frac{1}{x^3}, & x \geq 1 \end{cases}$$

a) F è continua, assolutamente crescente e
 $\lim_{x \rightarrow -\infty} F(x) = 0$, $\lim_{x \rightarrow +\infty} F(x) = 1$

$f(x) = F'(x)$ dove F è derivabile

$$f(x) = \begin{cases} 0, & x < 1 \\ 3 \frac{1}{x^4}, & x > 1 \end{cases}$$

$$E[X^k] = \int_{-\infty}^{+\infty} x^k f(x) dx = \int_1^{+\infty} 3 x^{k-4} dx < +\infty$$

$$x \geq 0 \Rightarrow k-4 < -1 \Leftrightarrow k < 3$$

$$E[X] = \int_1^{+\infty} 3 x^{-3} dx = \frac{3}{2}$$

$$E[X^2] = \int_1^{+\infty} 3 x^{-2} dx = 3$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{3}{4}$$

$E[X^3]$ non esiste

b) X, Y i.e. indip ed equidistribuite

$$E[(X-Y)^2] = \text{Var}(X-Y) + E[X-Y]^2$$

$$E[X-Y] = E[X] - E[Y] = 0$$

$$\begin{aligned} \text{Var}(X-Y) &= \text{Var}(X) + \text{Var}(-Y) = \text{Var}(X) + \text{Var}(Y) \\ &= 2 \text{Var}(X) \end{aligned}$$

$$E[(X-Y)^2] = \frac{3}{2}$$

c) X_1, \dots, X_{80} indipendenti e identiche.

$$P\{X_1 + \dots + X_{80} \leq 100\} = P\left\{\frac{X_1 + \dots + X_{80} - 120}{\sqrt{60}} \leq \frac{100 - 120}{\sqrt{60}}\right\}$$

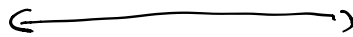
$$\frac{X_1 + \dots + X_{80} - 80 E[X_i]}{\sqrt{80 \text{Var}(X_i)}} \sim N(0, 1) = Z$$

$$80 E[X_i] = 120$$

$$80 \text{Var}(X_i) = 60$$

$$\sim P\left\{Z \leq -\frac{20}{\sqrt{60}}\right\} = \Phi\left(-\frac{10}{\sqrt{15}}\right) (= 0.00491 \text{ R})$$

$$\begin{aligned} \sim \Phi(-2.58199) &= 1 - \Phi(2.58199) \sim 1 - 0.995 \\ &= 0.005 \end{aligned}$$



ES 3 X_1, \dots, X_{142} $X_k \sim B(1, p)$ $\hat{p} = 0.23$

a) Intervallo di fiducia bilaterale per p

$$I = \left[\hat{p} - q_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + q_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]$$

$$1-\alpha = 0.9 \Leftrightarrow \alpha = 0.1 \Leftrightarrow 1-\frac{\alpha}{2} = 0.95$$

$$n = 192 \quad \hat{p} = 0.23$$

Precisione della stima

$$q_{0.95} \sim 1.645 \quad (= 1.644854 R)$$

$$q_{0.95} \sqrt{\frac{0.23(1-0.23)}{192}} \sim 0.0499$$

$$b) \quad H_0) p \geq 0.25 = p_0 \quad H_1) p < 0.25$$

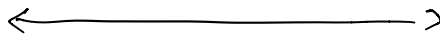
Test unilaterale per p al livello α

$$\text{Regione critica} \quad C = \left\{ \frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}} (\hat{p} - p_0) < q_\alpha \right\}$$

$$p\text{-value} \quad \alpha = \Phi \left(\frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}} (\hat{p} - p_0) \right) = \Phi(-0.64)$$

$$(= 0.26108 R)$$

$$= 1 - \Phi(0.64) \sim 1 - 0.73891 = 0.26109$$



ES.3 (vecchio programma)

Catene di Markov su $S = \{1, 2, 3, 4\}$

$$P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

a) $X = (a \ b \ c \ d)$ f.c. $a, b, c, d \geq 0$

$$\begin{cases} a+b+c+d = 1 \\ X P = X \end{cases}$$

$$\begin{cases} a+b+c+d = 1 \\ \frac{1}{4}a + \frac{1}{4}b + \frac{1}{2}c = a \\ \frac{1}{2}b + \frac{1}{3}d = b \\ \frac{3}{4}a + \frac{1}{4}b + \frac{1}{2}c + \frac{1}{3}d = c \\ \frac{1}{3}d = d \end{cases} \iff \begin{cases} a+c = 1 \\ \frac{1}{4}a + \frac{1}{2}c = a \\ b = 0 \\ \frac{3}{4}a + \frac{1}{2}c = c \\ d = 0 \end{cases}$$

$$\iff a = \frac{2}{5}, \quad c = \frac{3}{5}$$

$$X = \left(\frac{2}{5} \ 0 \ \frac{3}{5} \ 0 \right)$$

b) $X_0 = (0 \ 0 \ 0 \ 1)$

$$P\{X_2 = 3\} = \sum_{j=1}^4 \overbrace{P\{X_2 = 3 \mid X_1 = j\}}^{= P_{j3}} P\{X_1 = j\}$$

$$X_1 = X_0 P = \left(0 \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \right)$$

$$P\{X_2=3\} = \frac{1}{12} + \frac{1}{6} + \frac{1}{9} = \frac{13}{36}$$